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A Review on Stability Analysis of Continuous-Time Fuzzy-Model-Based Control Systems: From Membership-Function-Independent to Membership-Function-Dependent Analysis

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Abstract

This paper reviews the stability analysis of continuous-time fuzzy-model-based (FMB) control systems, with emphasis on state-feedback control techniques, which is an essential issue received a great deal of attention in the fuzzy control community. It gives an overview of the milestones and the trend of developments and achievements for the past decades. Focusing on the stability analysis of FMB control systems, it summarizes the issues in the four fundamental and essential aspects, namely, the types of membership-function matching, types of Lyapunov functions, types of stability analysis and the techniques of stability analysis are discussed. To start with systematic discussion, the FMB control systems are categorized into three types of membership-function matching, namely, perfectly, partially and imperfectly matched premises, regarding the premise membership functions and the number of rules used in the fuzzy model and fuzzy controller which forms the FMB control system. The features of each category are thoroughly discussed from theoretical to practical point of view. Various types of Lyapunov functions available in the literature for conducting stability analysis and their characteristics are then reviewed. Especially, the focus of this paper is to promote the concept of membership-function-dependent (MFD) stability analysis, which makes use of the information of membership func-

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tions aiming to relax the stability conditions compared with the dominant membership-function-independent (MFI) stability analysis in the literature. The techniques for MFI and MFD stability analysis are then discussed in details, which provide some solid ideas to analyze the stability of FMB control systems. More importantly, it sheds light on the fact that MFD stability analysis demonstrates a greater potential than the MFI one for relaxing the conservativeness of stability analysis results, which points a promising research direction for this topic. The purposes of this paper are to provide a comprehensive update for the stability analysis of FMB control systems to the researchers in the field and serve as a quick guide for the potential researchers who want to enter the field.

Keywords: Continuous-Time, Fuzzy-Model-Based (FMB) Control, Membership-Function-Dependent (MFD), Membership-Function-Independent (MFI), Perfectly\Partially\Imperfectly Matched Premises, Review, State Feedback, Stability Analysis

1. Introduction

Since the introduction of fuzzy sets in 1965 by Prof. Lotfi A. Zadeh [1], it has been rapidly developed to a promising research. By using linguistic rules, human spirit and knowledge can be incorporated into a fuzzy logic system. With the support of fuzzy set theory and mathematics, a fuzzy logic system can perform reasoning according to the designated linguistic rules. Fuzzy logic systems were used successfully in a wide range of areas and applications [2, 3, 4] such as assessment [5], classification [6, 7, 8], control [9, 10, 11], decision making [12, 13, 14, 15, 16, 17], evaluation [18, 19], forecasting [20, 21, 22], learning [23, 24, 25], modeling [26, 27] and etc.

Fuzzy control is a hot research topic, which has drawn a great deal of attention from researchers working from fundamental research to domestic/industrial applications in the past few decades. It has been witnessed that fuzzy control has developed from model-free approach [28] to model-based approach [29, 30, 31], from the author's view with focus on continuous-time systems and state-feedback control technique, which can be summarized by four stages as shown in the upper half section of Fig. 1 and some milestones happened during the development are marked in the lower half section of Fig. 1.

Stage 1 happened during early 1970s to late 1980s, where the concept of fuzzy control has been developed. Mamdani-type fuzzy logic controller [32, 33, 34] is an example, which consists of four basic units namely fuzzifier, rule base, inference engine and defuzzifier. It collects expert knowledge from the control problem and incorporates them using fuzzy sets and linguistic rules. Thanks to the fuzzy logic, the fuzzy logic controller can mimic the human spirit to reproduce the human control actions. As a result, it demonstrates that the fuzzy logic controller can handle ill-defined and complex nonlinear control problems well without the need of any mathematical models. So it is termed as a model-free control approach. Since then, a lot of successful applications have been reported such as sludge wastewater treatment [35] and control of cement kiln [36] found in 1980s, and then regulation of DC-DC power converters [37, 38], motor control [39] in 1990s. However, the model-free approach suffers from 1) the design is heuristic which is a time-consuming design process and may lead to inconsistent performance, 2) the system stability and robustness are not guaranteed but tested experimentally.

Stage 2 started mainly from early 1990s to mid-2000s, where the fuzzy-model-based (FMB) control concept kicked in. The Takagi-Sugeno (T-S) or also known as Takagi-Sugeno-Kang (T-S-K) fuzzy model [40, 41] plays an important role to support the system analysis and control design. It describes the dynamics of nonlinear system as an average weighted sum of some local linear sub-systems where the weights characterized by membership functions measure the contribution made by each. In general, there are three main approaches for constructing a T-S fuzzy model: 1) Applying system identification techniques to experimental data [40, 41], 2) Applying sector nonlinear techniques [42] to mathematical model, 3) Approximating the nonlinear system by combining linearized models at the chosen operating points with membership functions [43].

By connecting a state-feedback fuzzy controller [44, 45, 46, 47] (without otherwise stated, referred to as fuzzy controller hereafter) to a nonlinear plant represented by the T-S fuzzy model in a closed loop, an FMB based control system is formed. As the fuzzy controller is represented as an average weighted sum of linear state-feedback sub-controllers, the FMB control system is expressed as an average weighted sum of linear control sub-systems formed by the local linear sub-systems from the T-S fuzzy model and the linear state-feedback sub-controllers from the fuzzy controller.

In stage 2, the stability of FMB control systems were studied through mainly investigating the linear control sub-systems. Basic stability conditions in terms of linear matrix inequalities (LMIs) [48, 49] were obtained, where a feasible solution (if any) can be found numerically using convex programming techniques [44, 45]. Sector nonlinearity technique was then proposed to provide a systematic approach to construct a T-S fuzzy model for the nonlinear system based on its mathematical model [46, 47]. An important parallel distributed compensation (PDC) concept [46, 47] was proposed to relax the stability analysis results. The PDC design concept suggests that the fuzzy controller shares the same premise membership functions and the same number of rules from the T-S fuzzy model, which facilitates the stability analysis by grouping the same cross terms of membership functions possessed by the linear control sub-systems. Along the same line of analysis, further relaxed results were reported in [50, 51, 52, 53, 54] by using different grouping ways of cross terms and with the introduction of slack matrices and then generalized in [55] by considering the permutations of membership functions using Pólya theorem. Further follow-up work based on Pólya theorem can be found in [56, 57]. The aforementioned PDC results were obtained based on a common quadratic Lyapunov function, a non-PDC design concept was proposed in [58] where the Lyapunov function depending on the membership functions was used.

Some more stability analysis work based on T-S fuzzy model has continued after the period of stage 2 but it is less active comparatively. It is worth mentioning that the introduction of sum-of-squares (SOS) concept [59] inspires the development of polynomial fuzzy-model-based (PFMB) control systems [60, 61, 62]. A polynomial fuzzy model was proposed in [60], which extends the T-S fuzzy model to represent a wider class of nonlinear plants by allowing polynomials in the local sub-systems. Along the line of PDC design concept, stability conditions in terms of SOS were obtained in [60]. Since then, a lot of research on stability analysis of PFMB control systems have been carried on and variations of SOS-based stability conditions have been obtained for different control problems, just to name a few, such as observer-based control problems [63, 64, 65, 66, 67], output-feedback control problems [68], positive control problems [69], regulation control problems [70], sampled-data control problems [71], stabilization control problems [60, 72, 61, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 11, 87, 88], switching control problems [79], tracking control problems [89] and etc.

Generally speaking, the T-S FMB control system separates the closed-loop control system into the linear (linear control sub-systems) and nonlinear (membership functions) parts. Consequently, the stability analysis, which can be made easy by mainly investigating the linear part through the permutations of membership functions, are considered under the PDC design concept. However, the information of membership functions is not considered in the stability analysis and neither in the stability conditions. It is thus the author terms this stability analysis to be membership-function-independent (MFI). As the stability conditions are MFI, it means that the MFI stability analysis is for a family of FMB control systems with the same set of linear control sub-systems but any membership functions, which explains the source of conservativeness. The PDC is a dominant concept in the MFI stability analysis due to the permutations of the cross terms of membership functions can facilitate the analysis. However, the PDC design concept requires that the fuzzy model and fuzzy controller share the same sets of premise membership functions and the same number of rules, which impose limitations on various issues such as design flexibility, computational demand, implementation complexity, robustness property, analysis feasibility and applicability. The author advocates the concept of perfectly, partially and imperfectly matched premises [10], which categories the FMB control systems into three categories according to the premise membership functions and number of rules used in the fuzzy model and fuzzy controller. The three categories cannot be superseded by one or another due to each category demonstrates its own limitations and benefits. Membership-function-dependent (MFD) stability analysis makes possible and unifies the stability analysis for these three categories of FMB control systems. Also, utilizing the information of membership functions in the stability analysis provide an efficient way to offer more relaxed stability analysis results.

Stages 3 and 4 move from MFI stability analysis to MFD stability analysis where stage 3, happening from mid-2000s to late 2000s, focuses on FMB control systems, while stage 4, starting from late 2000s, focuses on PFMB control systems that polynomial fuzzy model [60, 61] is employed to represent the nonlinear plant. In stage 3, the author's work [90] attempted in the first time to bring the information of membership functions into stability analysis. Consequently, the stability conditions are MFD, where the conservativeness is alleviated compared with the MFI ones. Since then, MFD stability analysis results using various membership function information and techniques can be found for the FMB control systems [91, 92, 93, 75, 94, 95, 96, 97, 98] and the PFMB control systems [76, 77, 79, 82, 85]. The MFI stability analysis has also extended to interval type-2 (IT2) FMB/PFMB control systems [99, 100, 101, 102, 103, 104, 105, 106]. Further details regarding MFD stability analysis will be provided in the later sections.

The advancement of stability analysis on FMB/PFMB control systems supports extensively the development of various FMB/PFMB control systems [11], where some examples are shown in Fig. 2. It should be noted that Fig. 2 does not attempt to provide a complete picture showing all FMB/PFMB control systems in the literature but gives some popular examples. Detailed descriptions can be found in [11] and the references therein. In addition, the above discussion is only for the FMB/PFMB control systems using type-1 fuzzy sets, so they are referred to as type-1 FMB/PFMB control systems. In the literature, interval type-2 (IT2) fuzzy sets [107, 108] have been used to capture uncertainties which inspire the development of IT2 FMB/PFMB control systems. An IT2 fuzzy model was introduced in [99] to describe the dynamics of the nonlinear plant subject to uncertainties, where the uncertainties are captured by the footprint of uncertainty (FOU) characterized by the lower and upper membership functions. As a result of the membership grades of the IT2 fuzzy model are uncertain in value, the PDC design concept cannot facilitate the stability analysis any more. Stability analysis of IT2 FMB control systems has been first attempted in [99]. Since then, improvements and variations can be found in the literature [109, 100, 101, 110, 111, 102, 103, 112, 113, 105].

Fig. 3 gives an overview of the design process of FMB/PFMB control systems (both type-1 and IT2), which consists of four processes, namely, system modeling, controller design, stability analysis and control synthesis, and system control.

System modeling is the first step which constructs a fuzzy model such as T-S fuzzy model or polynomial fuzzy model to describe the dynamics of the nonlinear plant and capture its characteristic. Depending on the nature of the nonlinear plant, variations of T-S fuzzy model/polynomial fuzzy model can be employed to capture the characteristic, just to a name a few, such as disturbance [114, 88, 115, 116], input nonlinearity/saturation [117, 85, 118, 119, 120], jumping [121], positivity [122, 123, 124, 125, 69], stochastic processes [126, 127], time delay [128, 129, 130], switching [131, 79, 132, 133, 134], uncertainties [46, 135, 87], etc., in continuous-time/discrete-time form.

A fuzzy controller is then designed to close the feedback loop of the nonlinear plant. Variations of fuzzy controller have been proposed and some examples are given in Fig. 4, where the details can be found in [11] and the references therein. It should be noted that Fig. 4 does not attempt to provide a complete picture showing all fuzzy controllers in the literature but gives some popular examples.

An FMB/PFMB control system is formed by connecting the fuzzy model and fuzzy controller in a closed loop. Stability analysis can be conducted, for example, based on Lyapunov stability theorem, to obtain stability conditions in, for example, LMI/SOS forms. If there exists a feasible solution, the feedback gains of the fuzzy controller can be obtained and the stability of FMB/PFMB control system is guaranteed. The last process is to realize the fuzzy controller and perform system control. For physical applications, due to for example the disturbances, uncertainties and modeling error, employment of an appropriate fuzzy model and adjustment of feedback gains based on engineering sense may be required to make it work.

This paper focuses on reviewing the stability analysis of continuous-time FMB/PFMB control systems using state-feedback control techniques covering both MFI to MFD analysis concepts. The organization is as follow. In Section 2, the notations used are introduced. In Section 3, the fuzzy model, fuzzy controller and FMB/PFMB control system are reviewed. In Section 4, the stability analysis is reviewed and discussed focusing on four aspects, namely, the types of membership-function matching (perfectly, partially and imperfectly matched premises), types of Lyapunov functions candidates, types of stability analysis (MFI and MFD) and techniques of stability analysis, which are the major components for stability analysis and relaxation of stability conditions. Some issues related to bringing the theoretical results to practical applications are briefly discussed in Section 5. A brief discussion on future research directions is given in Section 6. In Section 7, a conclusion is drawn.

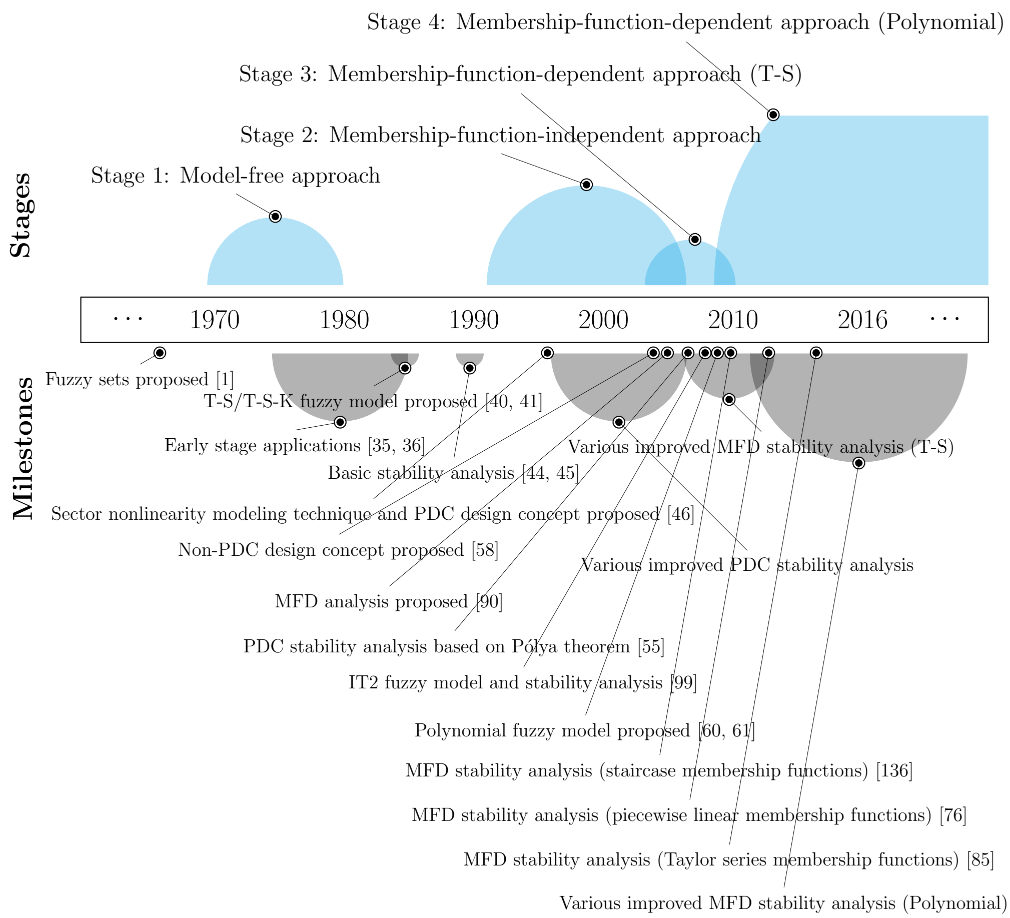

Figure 1: Stages and milestones of the research development of fuzzy-model-based control systems.

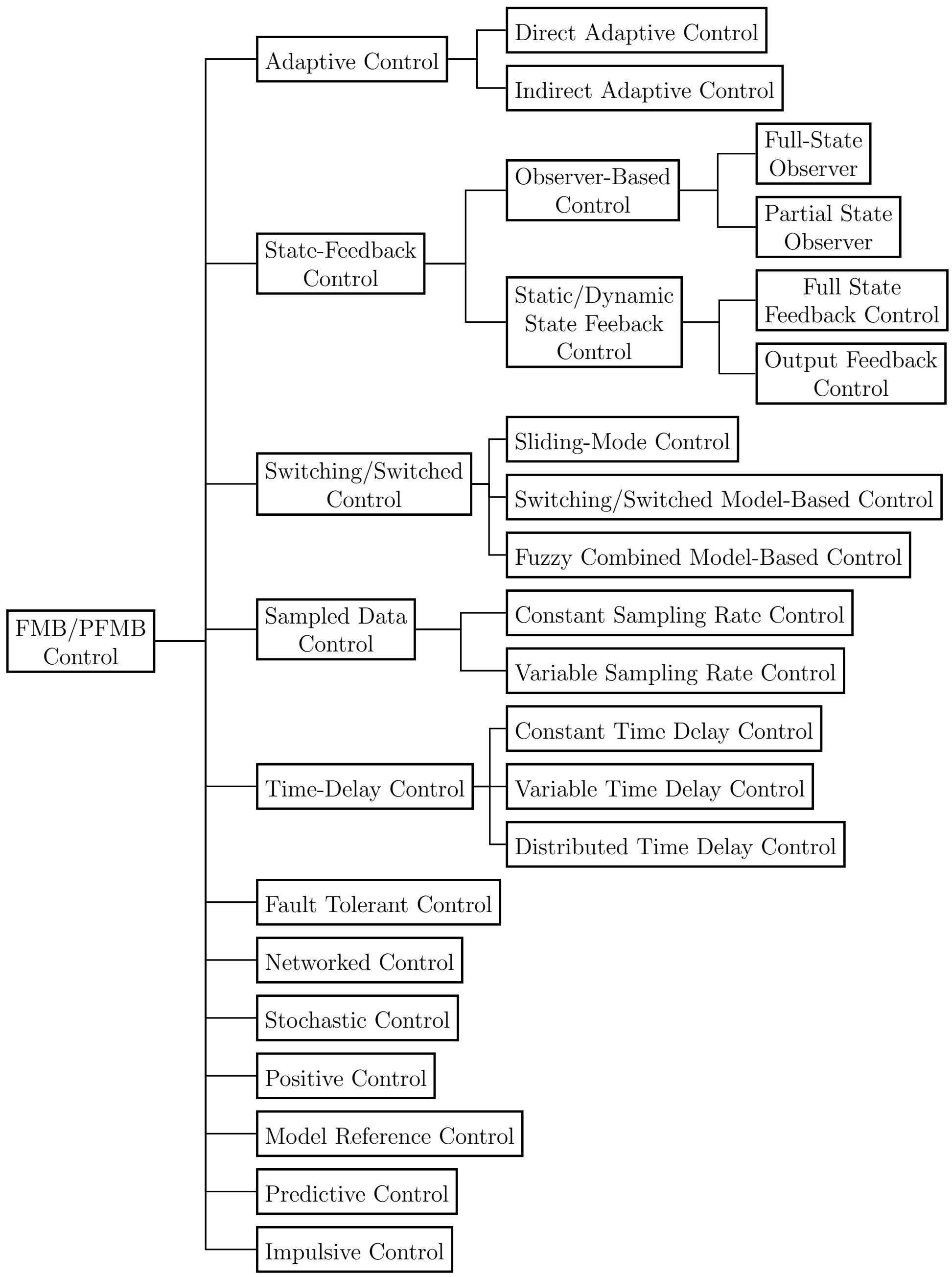

Figure 2: Various types of FMB/PFMB control strategies [11].

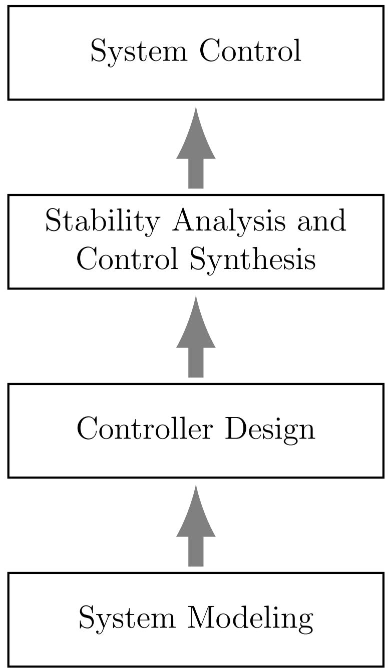

Figure 3: Design process of FMB/PFMB system [11].

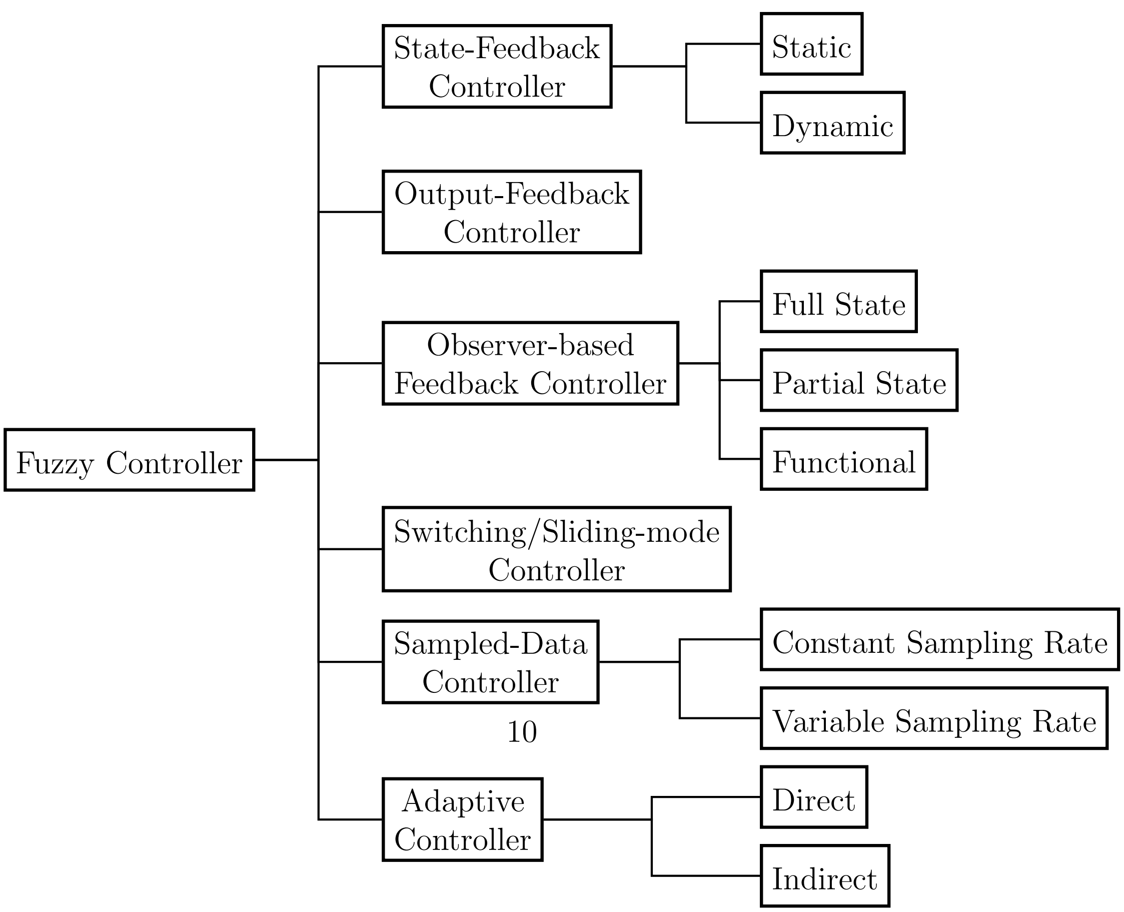

Figure 4: Various types of fuzzy controllers [11].

2. Notation

We adopt the following notations [59] throughout this paper. The monomial in $\mathbf{x}(t) = [x_1(t), \cdots, x_n(t)]^T$ is defined as $x_1^{d_1}(t) \cdots x_n^{d_n}(t)$, where d_i, $i = 1, \ldots, n$, are non-negative integers and the degree of a monomial is defined as $d = \sum_{i=1}^{n} d_i$. A polynomial $\mathbf{p}(\mathbf{x}(t))$ is defined as a finite linear combination of monomials with real coefficients. A polynomial $\mathbf{p}(\mathbf{x}(t))$ is an SOS if $\mathbf{p}(\mathbf{x}(t)) = \sum_{j=1}^{m} \mathbf{q}_j(\mathbf{x}(t))^2$ can be expressed, where $\mathbf{q}_j(\mathbf{x}(t))$ is a polynomial and m is a non-zero positive integer. Hence, $\mathbf{p}(\mathbf{x}(t)) \geq 0$ can be concluded if it is an SOS. The expressions of $\mathbf{M} > 0$, $\mathbf{M} \geq 0$, $\mathbf{M} < 0$ and $\mathbf{M} \leq 0$ denote the positive, semi-positive, negative, semi-negative definite matrices $\mathbf{M}$, respectively.

Remark 1. *The polynomial $\mathbf{p}(\mathbf{x}(t))$ being an SOS can be represented in the form of $\hat{\mathbf{x}}(t)^T \mathbf{Q} \hat{\mathbf{x}}(t)$, where $\hat{\mathbf{x}}(t)$ is a vector of monomials in $\mathbf{x}$ and $\mathbf{Q}$ is a positive semi-definite matrix [137]. The problem of finding a $\mathbf{Q}$ can be formulated as a semi-definite program (SDP). SOSTOOLS [138] is a third-party Matlab toolbox for solving SOS programs and its technical details can be found in [139].*

3. Preliminaries

An FMB control system is formed by connecting a fuzzy model and a fuzzy controller in a closed loop as shown in Fig. 5. When a polynomial fuzzy model is employed, the closed-loop control system becomes a PFMB control system. Referring to this figure, $\mathbf{x}(t) \in \Re^n$ is the system state vector, t denotes the time, $n > 0$ is an integer denoting the system order, $\mathbf{u}(t) \in \Re^m$ is the input vector, $m > 0$ is an integer denoting the number of inputs, $\mathbf{r}(t) \in \Re^n$ denotes the input command, $\mathbf{e}(t) \in \Re^n$ which is defined as $\mathbf{e}(t) = \mathbf{r}(t) - \mathbf{x}(t)$.

Without loss of generality, we consider $\mathbf{r}(t) = \mathbf{0}$ which reduces the control problem to stabilization problem that the control objective is to stabilize the FMB control system. To be more specific, in this paper, we consider asymptotic stability. It is, by properly designing a fuzzy controller, to drive the system states $\mathbf{x}(t)$ towards zero when time t tends to infinity.

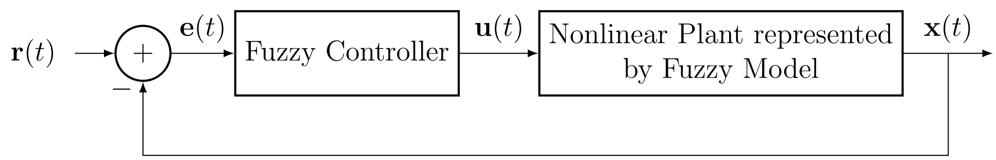

Figure 5: A block diagram of FMB/PFMB control system.

3.1. Polynomial Fuzzy Model

There are in general two types of fuzzy model, namely T-S fuzzy model [40, 41] and polynomial fuzzy model [60, 61], describing the dynamics of the nonlinear plant. A T-S fuzzy model can be considered as a reduced version of polynomial fuzzy model.

A polynomial fuzzy model describes the dynamics of the nonlinear plant using p rules of the following format:

$$\begin{aligned}&\text{Rule } i\text{: IF } f_1(\mathbf{x}(t)) \text{ is } M_1^i \text{ AND } \cdots \text{ AND } f_\Psi(\mathbf{x}(t)) \text{ is } M_\Psi^i \\ &\qquad \text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t), i = 1, \ldots, p,\end{aligned} \tag{1}$$

where M_α^i is a fuzzy set of rule i corresponding to the function $f_\alpha(\mathbf{x}(t))$, $\alpha = 1, \ldots, \Psi$; $i = 1, \ldots, p$; Ψ is a non-zero positive integer; $\mathbf{x}(t) \in \Re^n$ is the system state vector; $\mathbf{A}_i(\mathbf{x}(t)) \in \Re^{n\times N}$ and $\mathbf{B}_i(\mathbf{x}(t)) \in \Re^{n\times m}$ are the known polynomial system and input matrices, respectively; N is a non-zero positive integer; $\hat{\mathbf{x}}(\mathbf{x}(t)) \in \Re^N$ is a vector of monomials in $\mathbf{x}(t)$ and $\mathbf{u}(t) \in \Re^m$ is the input vector.

Remark 2. *It is assumed that $\hat{\mathbf{x}}(\mathbf{x}(t)) = \mathbf{0}$ if and only if $\mathbf{x}(t) = \mathbf{0}$. Consequently, driving $\hat{\mathbf{x}}(\mathbf{x}(t))$ to $\mathbf{0}$ is equivalent to driving $\mathbf{x}(t)$ to $\mathbf{0}$, which is considered in the stability analysis.*

The polynomial fuzzy model is defined as:

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))\big(\mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t)\big), \tag{2}$$

where

$$w_i(\mathbf{x}(t)) \geq 0 \ \forall\, i, \sum_{i=1}^{p} w_i(\mathbf{x}(t)) = 1, \tag{3}$$

$$w_i(\mathbf{x}(t)) = \frac{\prod_{l=1}^{\Psi} \mu_{M_l^i}(f_l(\mathbf{x}(t)))}{\sum_{k=1}^{p} \prod_{l=1}^{\Psi} \mu_{M_l^k}(f_l(\mathbf{x}(t)))} \;\forall\, i, \tag{4}$$

$w_i(\mathbf{x}(t))$, $i = 1, \ldots, p$, is the normalized membership grade; $\mu_{M_l^i}(f_l(\mathbf{x}(t)))$, $l = 1, \ldots, \Psi$, is the membership function corresponding to the fuzzy set M_l^i.

Remark 3. *The polynomial fuzzy model* (2) *is reduced to a T-S fuzzy model when* $\hat{\mathbf{x}}(\mathbf{x}(t)) = \mathbf{x}(t)$*, and* $\mathbf{A}_i(\mathbf{x}(t))$ *and* $\mathbf{B}_i(\mathbf{x}(t))$ *are constant matrices for all* i*.*

3.2. Polynomial Fuzzy Controller

A polynomial fuzzy controller [76, 75] is described by c rules of the following format:

$$\begin{aligned} &\text{Rule } j\text{: IF } g_1(\mathbf{x}(t)) \text{ is } N_1^j \text{ AND } \cdots \text{ AND } g_\Omega(\mathbf{x}(t)) \text{ is } N_\Omega^j \\ &\qquad \text{THEN } \mathbf{u}(t) = \mathbf{G}_j(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)), j = 1, \ldots, c, \end{aligned} \tag{5}$$

where N_β^j is a fuzzy set of rule j corresponding to the function $g_\beta(\mathbf{x}(t))$, $\beta = 1, \ldots, \Omega$; $j = 1, \ldots, c$; Ω is a positive integer; $\mathbf{G}_j(\mathbf{x}(t)) \in \Re^{m\times N}$, $j = 1, \ldots, c$, is the polynomial feedback gain to be determined.

The polynomial fuzzy controller is defined as,

$$\mathbf{u}(t) = \sum_{j=1}^{c} m_j(\mathbf{x}(t))\mathbf{G}_j(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)), \tag{6}$$

where

$$m_j(\mathbf{x}(t)) \geq 0 \;\forall\, j, \sum_{j=1}^{c} m_j(\mathbf{x}(t)) = 1, \tag{7}$$

$$m_j(\mathbf{x}(t)) = \frac{\prod_{l=1}^{\Omega} \mu_{N_l^j}(g_l(\mathbf{x}(t)))}{\sum_{k=1}^{c} \prod_{l=1}^{\Omega} \mu_{N_l^k}(g_l(\mathbf{x}(t)))} \;\forall\, j, \tag{8}$$

$m_j(\mathbf{x}(t))$, $j = 1, \ldots, c$, is the normalized membership grade; $\mu_{N_l^j}(g_l(\mathbf{x}(t)))$, $l = 1, \ldots, \Omega$, is the membership function corresponding to the fuzzy set N_l^j.

Remark 4. *The polynomial fuzzy controller* (6) *is reduced to the traditional fuzzy controller when the feedback gains* $\mathbf{G}_j(\mathbf{x}(t))$ *are constant matrices for all* j.

3.3. Polynomial Fuzzy-Model-Based Control System

By connecting the polynomial fuzzy model (2) and the polynomial fuzzy controller (6) in a closed loop as shown in Fig. 5, with the property of the membership functions in (3) and (7), i.e., $\sum_{i=1}^{p} w_i(\mathbf{x}(t)) = \sum_{j=1}^{c} m_j(\mathbf{x}(t)) = \sum_{i=1}^{p} \sum_{j=1}^{c} w_i(\mathbf{x}(t)) m_j(\mathbf{x}(t)) = 1$, we obtain the PFMB control system as follows:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \sum_{i=1}^{p} w_i(\mathbf{x}(t)) \big(\mathbf{A}_i(\mathbf{x}(t)) \hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t)) \sum_{j=1}^{c} m_j(\mathbf{x}(t)) \mathbf{G}_j(\mathbf{x}(t)) \hat{\mathbf{x}}(\mathbf{x}(t)) \big) \\ &= \sum_{i=1}^{p} \sum_{j=1}^{c} w_i(\mathbf{x}(t)) m_j(\mathbf{x}(t)) \big(\mathbf{A}_i(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t)) \mathbf{G}_j(\mathbf{x}(t)) \big) \hat{\mathbf{x}}(\mathbf{x}(t)). \end{aligned} \tag{9}$$

Remark 5. *When the stabilization control problem is considered, the control objective is to design the feedback gains* $\mathbf{G}_j(\mathbf{x}(t))$ *such that the PFMB control system* (9) *is asymptotically stable, i.e.,* $\mathbf{x}(t) \to \mathbf{0}$ *as time* $t \to \infty$.

4. Stability Analysis

Fig. 6 shows the four major aspects to be considered when investigating the stability of the FMB/PFMB control systems in the form of (9). These aspects, namely the types of membership functions, types of Lyapunov functions, types of stability analysis and techniques of stability analysis, play an important role in connection to the following issues of the FMB/PFMB control systems:

- Applicability: It is about the capability of applying the analysis results and design methods towards control problems.
- Computation: it is about the level of computational demand required to solve a feasible solution to the stability conditions obtained under the same approach of stability analysis.
- Conservativeness: It is about the conservative level of the stability conditions obtained under the same approach of stability analysis.

	Perfectly Matched Premises	Partial Matched Premises	Imperfectly Matched Premises
Applicability	Low	Medium	High
Complexity	High	Medium	Low
Computation	High	Medium	Low
Conservativeness	Low	Medium	High
Flexibility	Low	Medium	High
Implementation	High	Medium	Low
Robustness	Low	Medium	High

Table 1: Comparison of various issues among the three categories of matched premises.

- Complexity: It is about the structural complexity of the fuzzy controller.
- Flexibility: It is about the design flexibility of the fuzzy controller such as the freedom of choosing its membership functions and number of rules.
- Implementation: It is about the costs of realizing the fuzzy controller.
- Robustness: It is about the capability of the fuzzy controller to tolerate uncertainties of the nonlinear plant, which are embedded in the membership functions of the fuzzy model.

For brevity, in the following, the membership functions $w_i(\mathbf{x}(t))$ and $m_j(\mathbf{x}(t))$ are denoted as w_i and m_j, respectively.

4.1. Types of Membership-Functions Matching

Referring to the PFMB control system (9), three types of membership-function matching, namely perfectly matched premises, partially matched premises and imperfectly matched premises, are summarized in Fig. 7 according to the membership functions and number of rules used in the polynomial fuzzy model (2) and polynomial fuzzy controller (6).

The comparison among the three categories of FMB/PFMB control systems in terms of the aforementioned issues, is summarized in Table 1, where details are given in the following subsections.

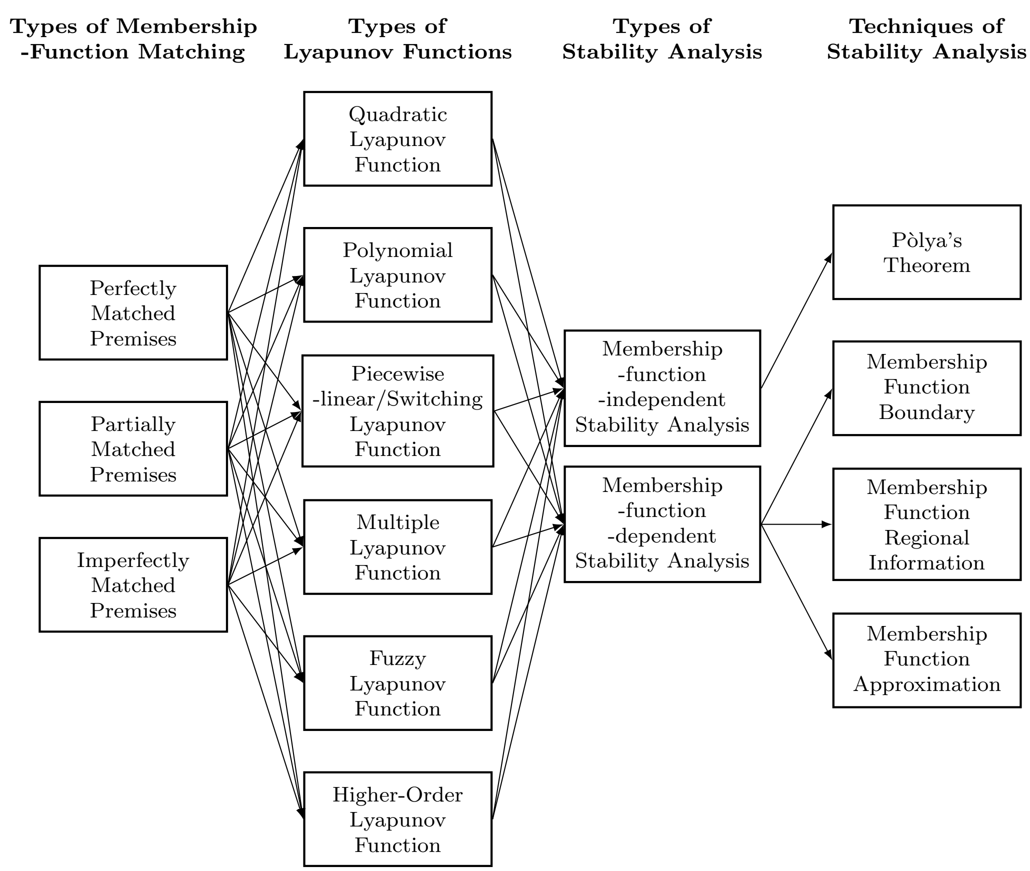

Figure 6: Types of membership-function matching, Lyapunov functions and stability analysis, and techniques of stability analysis.

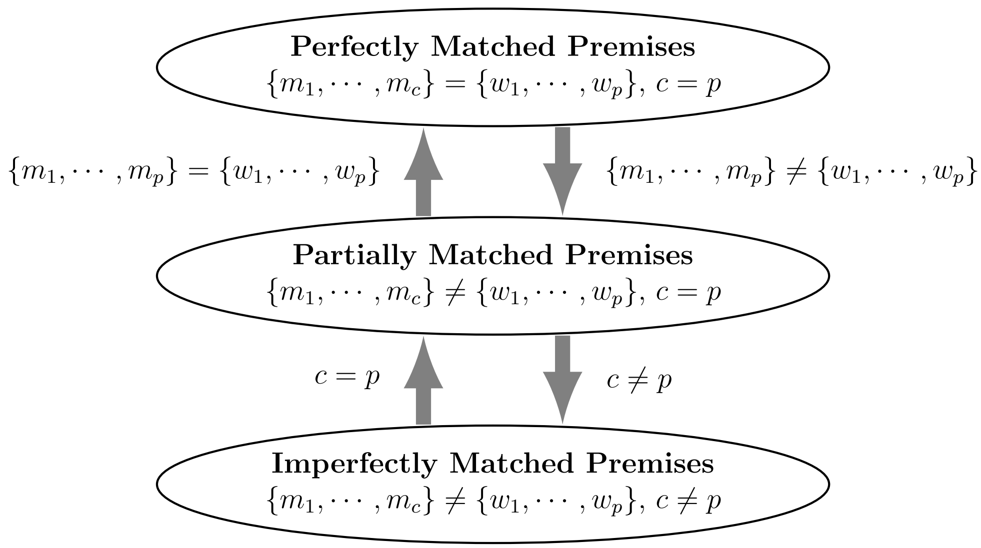

Figure 7: Three categories of FMB/PFMB control systems [11].

4.1.1. Perfectly Matched Premises

Under perfectly matched premises, the polynomial fuzzy model and polynomial fuzzy controller are required to share the same set of premise membership functions and the same number of rules, i.e., $\{m_1, \cdots, m_c\} = \{w_1, \cdots, w_p\}$ and $c = p$, which is also known well as PDC in the literature. Because of the matching of the premise membership functions, it is in favor of the stability analysis resulting in more relaxed stability conditions by properly grouping the cross terms of membership functions, i.e., $w_i m_j$, say, using Pólya theorem [55].

As the premise membership functions and the number of rules of the polynomial fuzzy controller are required to be the same as those of the polynomial fuzzy model, it constrains very much the design flexibility of the polynomial fuzzy controller. When the premise membership functions are complicated and/or the number of rules is large, it will increase the structural complexity of the polynomial fuzzy controller leading to the increase of implementation costs. Furthermore, it may reduce its applicability towards control applications as special shape of membership functions have to be implemented.

When the number of rules increases, the increase in the number of stability conditions subject to the same approach of stability analysis is related to p^2, which is relatively higher compared with the category of imperfectly matched premises. Consequently, in general, the computational demand required to solve a solution to the stability conditions under the category of perfectly matched premises is higher.

4.1.2. Partially Matched Premises

Under partially matched premises, the polynomial fuzzy controller does not require to share the same set of premise membership functions from the polynomial fuzzy model but the number of rules is required to be the same, i.e., $\{m_1, \cdots, m_c\} \neq \{w_1, \cdots, w_p\}$ and $c = p$. As the premise membership functions for the polynomial fuzzy controller can be freely chosen, some simple shape can be used to reduce the structural complexity of the polynomial fuzzy controller which can lower its implementation costs. Furthermore, it can improve its applicability to control applications as generic fuzzy logic controller can be used for implementation. Because the membership functions of the polynomial fuzzy model are not necessary to be known in the stability analysis and for the implementation of polynomial fuzzy controller, the polynomial fuzzy controller can handle well the nonlinear plant by embedding the uncertainties into the premise membership functions of its polynomial fuzzy model. Consequently, the polynomial fuzzy controller under partially matched premises demonstrates better robustness property towards the model uncertainties compared with the category of perfectly matched premises.

In general, compared with the category of perfectly matched premises, the stability analysis is less relaxed due to the mismatched membership functions as a result of PDC stability analysis technique cannot be directly applied. However, because the same number of rules are used, some techniques can be applied to facilitate the stability analysis by using the property of perfectly matched premises [75].

Similar to the category of perfectly matched premises, the increase in the number of rules will increase the number of stability conditions subject to the same approach of stability analysis related to p^2, which will lead to relatively higher computational demand to solve a solution to the stability conditions. Furthermore, some extra stability conditions are required to deal with the mismatched premise membership functions.

4.1.3. Imperfectly Matched Premises

Under imperfectly matched premises, the constrains on the premise membership functions and the number of rules for the polynomial fuzzy controller are removed, i.e., $\{m_1, \cdots, m_c\} \neq \{w_1, \cdots, w_p\}$ and $c \neq p$ are allowed. Thus, the polynomial fuzzy controller demonstrates the largest design flexibility for freely choosing the premise membership functions and the number of rules. However, it will lead to potentially more conservative stability analysis results compared with the categories of perfectly matched premises and partially matched premises, due to the PDC stability analysis technique (using the permutations of the cross terms of membership functions) cannot be applied. As the membership functions of the polynomial fuzzy model are not necessary to be known in the stability analysis and for the implementation of polynomial fuzzy controller, for the same reason given in the category of partially matched premises, the polynomial fuzzy controller under imperfectly matched premises demonstrates better robustness towards the model uncertainties compared with the category of perfectly matched premises.

By employing simple shape of membership functions and smaller number of rules, a simpler polynomial fuzzy controller can be implemented at lower costs, say, using a generic fuzzy controller, even for nonlinear plants with complex polynomial fuzzy models, which will thus enhance the applicability towards control applications.

As the number of rules between the polynomial fuzzy model and polynomial fuzzy controller can be different, the number of stability conditions is related to $p \times c$. By choosing a smaller number of rules for the polynomial fuzzy controller, subject to the same approach of stability analysis, it will lead to less number of stability conditions resulting in lower computational demand on solving a feasible solution (if any). However, some additional stability conditions have to be used to deal with the mismatched premise membership functions.

Remark 6. *From the research point of view, the partially and imperfectly matched premises address the fundamental issues of some emerging research topics of FMB/PFMB control systems, such as time-delayed [104], sampled-data [140, 68, 71], observer-based [141, 65], IT2 [105, 106], networked [142, 143, 144] fuzzy control systems. For all these control systems, the grade of membership functions are not the same as those of the fuzzy model due to the system states obtained by the fuzzy controller are altered, say, by the time delay, sampling process, or observer. For example, when networked control* *is considered, the system states obtained at the fuzzy controller is the time-delayed version due to packet drop out, delay time required for transferring signal to the remote side. Consequently, at the same time instance, the system states used in the membership functions of the fuzzy controller are not the same as the ones used in the fuzzy model. No matter the fuzzy model and fuzzy controller use the same set of premise membership functions or not, the grade of membership functions between them are not the same leading to the case of partially/imperfectly matched premises. The stability analysis techniques discussed in this paper regarding partially/imperfectly matched premises provide substantial fundamental support to the control problems demonstrating the aforementioned property, which receives rarely attention and achieves limited results in the field.*

4.2. Types of Lyapunov Function Candidates

Lyapunov stability theorem [145, 146] plays an important role in the stability analysis of FMB/PFMB control systems. It is described in brief that a Lyapunov function $V(\mathbf{x}(t))$ is a positive definite function which satisfies $V(\mathbf{0}) = 0$ and $V(\mathbf{x}(t)) > 0 \ \forall \ \mathbf{x}(t) \neq 0$. The equilibrium point $\mathbf{x}(t) = \mathbf{0}$ is asymptotically stable if $\dot{V}(0) = 0$ and $\dot{V}(\mathbf{x}(t)) < 0 \ \forall \ \mathbf{x}(t) \neq \mathbf{0}$ can be achieved.

Some Lyapunov function candidates found in the literature include quadratic Lyapunov function, polynomial Lyapunov function, piecewise-linear/switching Lyapunov function, fuzzy Lyapunov function and higher-order Lyapunov function [11]. Employing different Lyapunov function candidates for stability analysis will lead to different levels of relaxation of stability conditions. In general, more complex Lyapunov function candidates will usually lead to more relaxed stability conditions. However, it requires more advanced techniques to conduct stability analysis and will usually lead to more complex stability conditions.

4.2.1. Quadratic Lyapunov Function Candidate

A quadratic Lyapunov function candidate is defined as

$$V(\mathbf{x}(t)) = \hat{\mathbf{x}}(t)^T \mathbf{P} \hat{\mathbf{x}}(t) \tag{10}$$

which satisfies $V(\mathbf{0}) = 0$ and $V(\hat{\mathbf{x}}(t)) > 0 \ \forall \ \mathbf{x}(t) \neq \mathbf{0}$ where $\mathbf{x}(t) = \begin{bmatrix} x_1(t), & \cdots, & x_n(t) \end{bmatrix}^T$ denotes the system state vector, $\hat{\mathbf{x}}(\mathbf{x}(t)) = \begin{bmatrix} \hat{x}_1(\mathbf{x}(t)), & \cdots, & \hat{x}_N(\mathbf{x}(t)) \end{bmatrix}^T$

is a vector of monomials in $\mathbf{x}(t)$ and $0 < \mathbf{P} = \mathbf{P}^T = \begin{bmatrix} P_{11} & \cdots & P_{1N} \\ \vdots & \ddots & \vdots \\ P_{N1} & \cdots & P_{NN} \end{bmatrix}$ is a constant matrix.

Taking the first time derivative of $V(\mathbf{x}(t))$, we obtain

$$\dot{V}(\mathbf{x}(t)) = \dot{\hat{\mathbf{x}}}(t)^T \mathbf{P} \hat{\mathbf{x}}(t) + \hat{\mathbf{x}}(t)^T \mathbf{P} \dot{\hat{\mathbf{x}}}(t) < 0. \tag{11}$$

If $\dot{V}(\mathbf{x}(t)) < 0$ for all $\mathbf{x}(t) \neq \mathbf{0}$ is satisfied, the PFMB control system (9) is guaranteed to be asymptotically stable, i.e., $\mathbf{x}(t) \to \mathbf{0}$ as time $t \to \infty$.

The quadratic Lyapunov function candidate is simple and extensively used in the literature. However, it always leads to more conservative stability analysis results compared with other more sophisticated Lyapunov function candidates to be discussed in the following due to limited amount of characteristics of the PFMB control system is utilized.

4.2.2. Polynomial Lyapunov Function Candidate

A polynomial Lyapunov function candidate is a polynomial function of even degrees. When the degrees are reduced to 2, the polynomial Lyapunov function candidate is reduced to a quadratic one.

In general, a polynomial Lyapunov function candidate takes the form of $V(\mathbf{x}(t)) = \hat{\mathbf{x}}(\mathbf{x}(t))^T \mathbf{P}(\mathbf{x}(t)) \hat{\mathbf{x}}(\mathbf{x}(t)) \geq 0$ (equality holds for $\mathbf{x}(t) = \mathbf{0}$) where $0 < \mathbf{P}(\mathbf{x}(t)) = \mathbf{P}(\mathbf{x}(t))^T = \begin{bmatrix} P_{11}(\mathbf{x}(t)) & \cdots & P_{1N}(\mathbf{x}(t)) \\ \vdots & \ddots & \vdots \\ P_{N1}(\mathbf{x}(t)) & \cdots & P_{NN}(\mathbf{x}(t)) \end{bmatrix}$ is a polynomial matrix.

In the literature, the polynomial matrix $\mathbf{P}(\mathbf{x}(t))$ depending on all elements of $\mathbf{x}(t)$ will lead to non-convex stability conditions due to some terms generated by $\frac{d\mathbf{P}(\mathbf{x}(t))}{dt}$. A workaround is to replace $\mathbf{P}(\mathbf{x}(t))$ by $\mathbf{P}(\tilde{\mathbf{x}}(t))$ where $\tilde{\mathbf{x}}(t) = \begin{bmatrix} \tilde{x}_{k_1}(\mathbf{x}(t)), & \cdots, & \tilde{x}_{k_q}(\mathbf{x}(t)) \end{bmatrix}^T$, $k_1, \cdots, k_q$ denote the row numbers that the entries of the entire row of the input polynomial matrix $\mathbf{B}_i(\mathbf{x}(t))$ are all zero for all i [60]. However, using $\mathbf{P}(\mathbf{x}(t))$ has potential to relax the stability analysis as more state information is utilized. Two-step approach [147], solution search method [87], homogeneous polynomial form of Lyapunov function [148] and elimination technique by setting equality constraints [149] were proposed in the stability analysis using $\mathbf{P}(\mathbf{x}(t))$ as the polynomial Lyapunov function candidate.

4.2.3. Piecewise-Linear/Switching Lyapunov Function Candidate

Piecewise-linear [150, 151, 152, 153, 154, 155] and switching [131, 79, 132, 133, 134] Lyapunov function candidates consist of a number of local Lyapunov function candidates, for example, in quadratic form [150, 151] or polynomial form [79].

Monotonically decaying of local Lyapunov function candidates individually is not sufficient to prove the system stability as shown in Fig. 8. Assuming that the piecewise-linear/switching Lyapunov function candidate $V(\mathbf{x}(t))$ consists of 4 local Lyapunov function candidates $V_1(\mathbf{x}(t))$ to $V_4(\mathbf{x}(t))$ where the switching takes place at time t_1, t_2 and t_3. Although each local Lyapunov function candidate is monotonic decaying, the overall trend of $V(\mathbf{x}(t))$ is increasing. If this trend keeps going, $V(\mathbf{x}(t))$, for example the switching points '$\bullet$', will approach infinity implying an unstable system with unbounded system states, i.e., $\mathbf{x}(t) \to \infty$.

In [150, 151, 152, 153, 154, 131, 79], continued switching points are used to make sure monotonic decaying of individual local Lyapunov function candidates which also imply that the Lyapunov function candidate $V(\mathbf{x}(t))$ is monotonic decaying. This concept is illustrated in Fig. 9, which requires that the left and right switching points at the switching instant are the same. However, as the inverse of the Lyapunov function matrix will appear in the stability analysis and considered as a decision variable, some clever but complex methods [150, 151, 152, 153, 154, 131] have been proposed to make sure that the matrix inverse does not destroy the continuity at the switching points in the Lyapunov function candidate. In [79], a switching polynomial Lyapunov function candidate that the local Lyapunov functions are polynomial functions was proposed which gives a simple design to preserve the continuity at the switching points.

In the above discussion, switching is according to the state space. The operating domain in state space is divided into a number of operating subdomains and each is associated with a local Lyapunov function candidate. Mathematically, it can be described that the operating domain $\mathbf{\Phi}$ which is divided into D connected operating sub-domains Φ_d, $d = 1, \ldots, D$, i.e., $\Phi = \bigcup_{d=1}^{D} \Phi_d$. Corresponding to the i^{th} operating sub-domains, the corresponding i^{th} local Lyapunov function candidate $V_i(\mathbf{x}(t))$ will be used for stability analysis. The piecewise-linear/switching Lyapunov function candidate can be expressed as

$$V(\mathbf{x}(t)) = \begin{cases} V_1(\mathbf{x}(t)) & \text{for } \mathbf{x}(t) \in \Phi_1, \\ & \vdots \\ V_D(\mathbf{x}(t)) & \text{for } \mathbf{x}(t) \in \Phi_D. \end{cases} \quad (12)$$

Fig. 10 shows a more general concept that switching instants do not necessary to be continuous. To make sure that the trend of the piecewise-linear/switching Lyapunov function candidate is monotonic decaying, a sufficient condition is that the present left-hand-side switching point indicated by '•' is lower than the previous one.

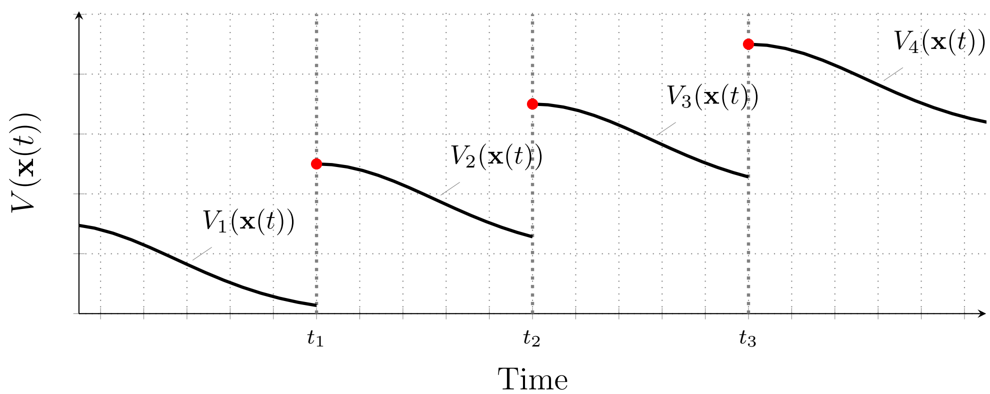

Figure 8: An example of an unstable switching Lyapunov function candidate, where '•' indicates the right-hand-side switching points [11].

4.2.4. Multiple Lyapunov Function Candidate

A multiple Lyapunov function candidate $V(\mathbf{x}(t))$ [156, 157] consists of a number of sub-Lyapunov function candidates, i.e., $V_1(\mathbf{x}(t)), \ldots, V_D(\mathbf{x}(t))$. All sub-Lyapunov function candidates are evaluated simultaneously and only one of them is picked at any instant to determine the system stability, for example, the minimum of them is picked, i.e., $V(\mathbf{x}(t)) = \min_{\mathbf{x}(t)} \{V_1(\mathbf{x}(t)), \ldots, V_D(\mathbf{x}(t))\}$.

Generally speaking, a multiple Lyapunov function candidate is a kind of switching Lyapunov function candidate that the switching instant is determined by the evaluated values of the sub-Lyapunov function candidates. As

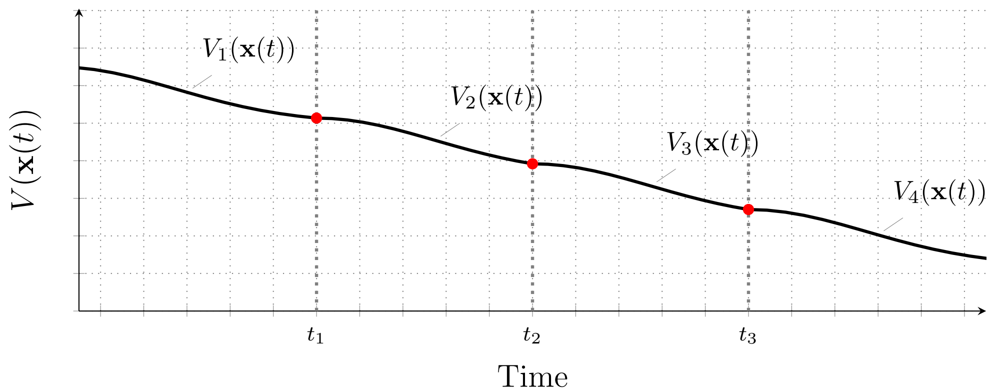

Figure 9: An example of a stable switching Lyapunov function candidate with continuous switching points indicated by '•' [11].

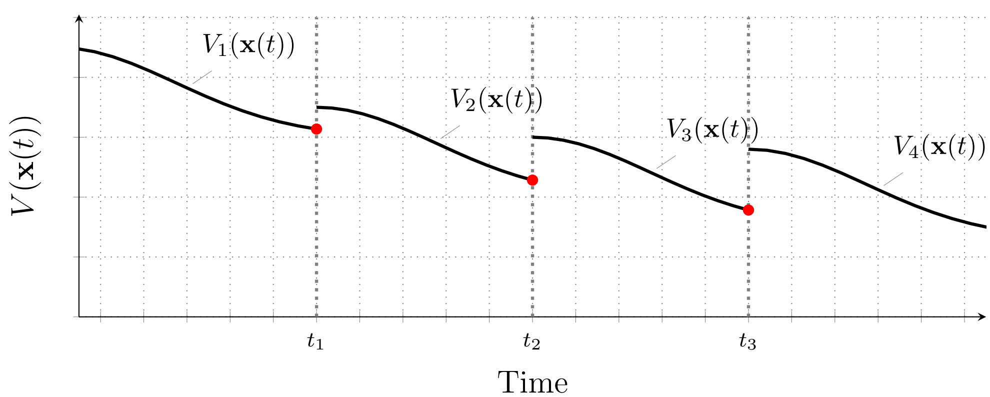

Figure 10: An example of a stable switching Lyapunov function candidate with lower switching points indicated by '•' compared with its previous one [11].

all the sub-Lyapunov function candidates are continuous functions, switching from one to another will maintain the continuity of the overall Lyapunov function candidate $V(\mathbf{x}(t))$ as shown in Fig. 9. One issue using the multiple Lyapunov function candidate is that it will lead to non-convex stability conditions due to S-procedure is applied. Some solution search techniques [156, 157] were proposed to find a feasible solution numerically.

4.2.5. Fuzzy Lyapunov Function Candidate

A fuzzy Lyapunov function candidate [158, 58, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168, 78, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181] is expressed as an average weighted sum of some local Lyapunov function candidates $V_k(\mathbf{x}(t))$.

The general form of the fuzzy Lyapunov function candidate can be expressed as follows:

$$V(\mathbf{x}(t)) = \sum_{k=1}^{q} n_k(\mathbf{x}(t)) V_k(\mathbf{x}(t)) \tag{13}$$

where $n_k(\mathbf{x}(t)) \geq 0$ is a normalized membership function which satisfies $\sum_{k=1}^{q} n_k(\mathbf{x}(t)) = 1$.

When taking time derivative on $V(\mathbf{x}(t))$, $\dot{n}_k(\mathbf{x}(t))$ will be produced and it will complicate the stability analysis. Various techniques for continuous-time case [161, 162, 163, 164, 165, 166, 182] and discrete-time case [58, 159] were proposed to deal with the issues caused by $\dot{n}_k(\mathbf{x}(t))$ and $n_k(\mathbf{x}(t)) - n_k(\mathbf{x}(t-1))$, respectively, in the stability analysis.

In the literature, $n_k(\mathbf{x}(t))$ is usually chosen to be the same membership function used in the T-S/polynomial fuzzy model under the PDC design concept. However, as discussed in Section 4.1, using a different set of premise membership functions in the fuzzy Lyapunov function candidate as in partially premise matching or imperfectly premise matching will benefit the fuzzy controller in terms of less complex structure, greater design flexibility and lower implementation costs (referring to Table 1). However, the stability analysis would potentially lead to relatively conservative stability conditions compared with the category of perfectly premise matching (or PDC design concept) due to mismatched premise membership functions. The work in [170, 11] proposed to use different sets of premise membership functions among the fuzzy model, fuzzy controller and/or the fuzzy Lyapunov function candidate (i.e., $w_i(\mathbf{x}(t))$, $m_j(\mathbf{x}(t))$ and $n_k(\mathbf{x}(t))$ are not necessary to be the same) and use the information of membership functions for relaxing the stability analysis results.

4.2.6. Higher-Order Lyapunov Function Candidate

A higher-order Lyapunov function candidate [183] consists of some higher-order derivative terms, which can be generally expressed as $V(\mathbf{x}(t)) = \sum_{k=0}^{q} \frac{d^k V_k(\mathbf{x}(t))}{dt}$ where $V_k(\mathbf{x}(t))$ is a scalar function and k denotes the order of derivative. When $q = 0$, the higher-order Lyapunov function candidate is reduced to any of the aforementioned Lyapunov function candidate.

A special case is that all $V_k(\mathbf{x}(t))$ are the same function for all k, say, $V_k(\mathbf{x}(t)) = V_0(\mathbf{x}(t)) \; \forall \; k$, where, for example, $V_0(\mathbf{x}(t))$ can be any of the above mentioned Lyapunov function candidates, resulting in $V(\mathbf{x}(t)) = \sum_{k=0}^{q} \frac{d^k V_0(\mathbf{x}(t))}{dt}$. It is required that $\dot{V}(\mathbf{x}(t))$ is negative definite to guarantee the system stability. With the higher-order derivatives, it implies that the system stability is not necessarily governed by requiring only $\dot{V}_0(\mathbf{x}(t))$ to be negative definite but the sum of all derivative terms. In other words, $V_0(\mathbf{x}(t))$ is not necessary to be monotonic decaying but the sum of all derivative terms is. Consequently, a higher-order Lyapunov function candidate demonstrates potential to produce relaxed stability analysis results. However, the derivative terms will complicate the stability analysis for FMB/PFMB control systems due to, for example, the derivative terms of membership functions produced.

In the literature of FMB/PFMB control systems, the concept of higher-order Lyapunov function candidate is applied mainly in discrete-time systems [184, 185, 186, 187, 188, 189, 190] but much less in continuous-time systems [191] as the derivative terms of membership functions are far more difficult to be handled in continuous-time systems.

4.3. Types of Stability Analysis

Referring to Fig. 6, there are mainly two types of stability analysis in terms of whether the information of membership functions being used, namely MFI and MFD stability analysis [11].

As MFI stability analysis does not take into account the information of membership functions, the stability conditions are for a family of fuzzy models sharing the same set of consequent rules, i.e., the THEN part in (1), regardless of the membership functions. The polynomial fuzzy controller obtained by finding a feasible solution to the MFI stability conditions can stabilize that whole family of FMB/PFMB control systems, which implies the conservativeness of the stability analysis.

In the MFD stability analysis, the stability conditions will contain some information of membership functions. Consequently, the stability conditions are MFD, which are more dedicated to the T-S/polynomial fuzzy model (representing the nonlinear plant) considered on hand depending on the level of the information of membership functions being used. The more the information of membership functions is considered, the more the stability conditions dedicate to the T-S/polynomial fuzzy model being considered.

Compared with the MFD stability analysis, MFI stability analysis is simpler, in terms of fewer number of stability conditions and decision variables, as the membership functions are usually dropped to obtain the stability conditions. In the MFD stability analysis, more stability conditions and decision variables are required to address the nonlinearity/characteristic of nonlinear plant (represented by a T-S/polynomial fuzzy model) and polynomial fuzzy controller embedded in the membership functions, resulting in more relaxed stability conditions.

In practice, to deal with control problem, the MFI stability conditions can be applied in the first place with the consideration that less computational demand is required to find a feasible solution. If no feasible solution is found for the MFI stability conditions, the MFD stability conditions can be employed starting with the minimum amount of information of membership functions for lower computational demand. When no feasible solution is found, more information of membership functions can be added to the MFD stability conditions for relaxing conservativeness.

4.4. Techniques of Stability Analysis

The techniques of stability analysis are discussed under MFI and MFD stability analysis with an overview shown in Fig. 11, which expands the "Techniques of Stability Analysis" in Fig. 6.

4.4.1. MFI Stability Analysis Techniques

The techniques for MFI stability analysis will drop the membership functions in the process of constructing stability conditions. For simplicity but without loss of generality, we consider the quadratic Lyapunov function (10) for the stability analysis of the PFMB control system (9). Taking the first time derivative of (10), it will lead to $\dot{V}(\mathbf{x}(t))$ in (11). Note that $\dot{\hat{\mathbf{x}}}(t)$ appearing in (11) can be expressed as $\dot{\hat{\mathbf{x}}}(t) = \frac{\partial \hat{\mathbf{x}}(t)}{\partial \mathbf{x}(t)} \dot{\mathbf{x}}(t)$. Substituting $\dot{\mathbf{x}}(t)$ in (9)

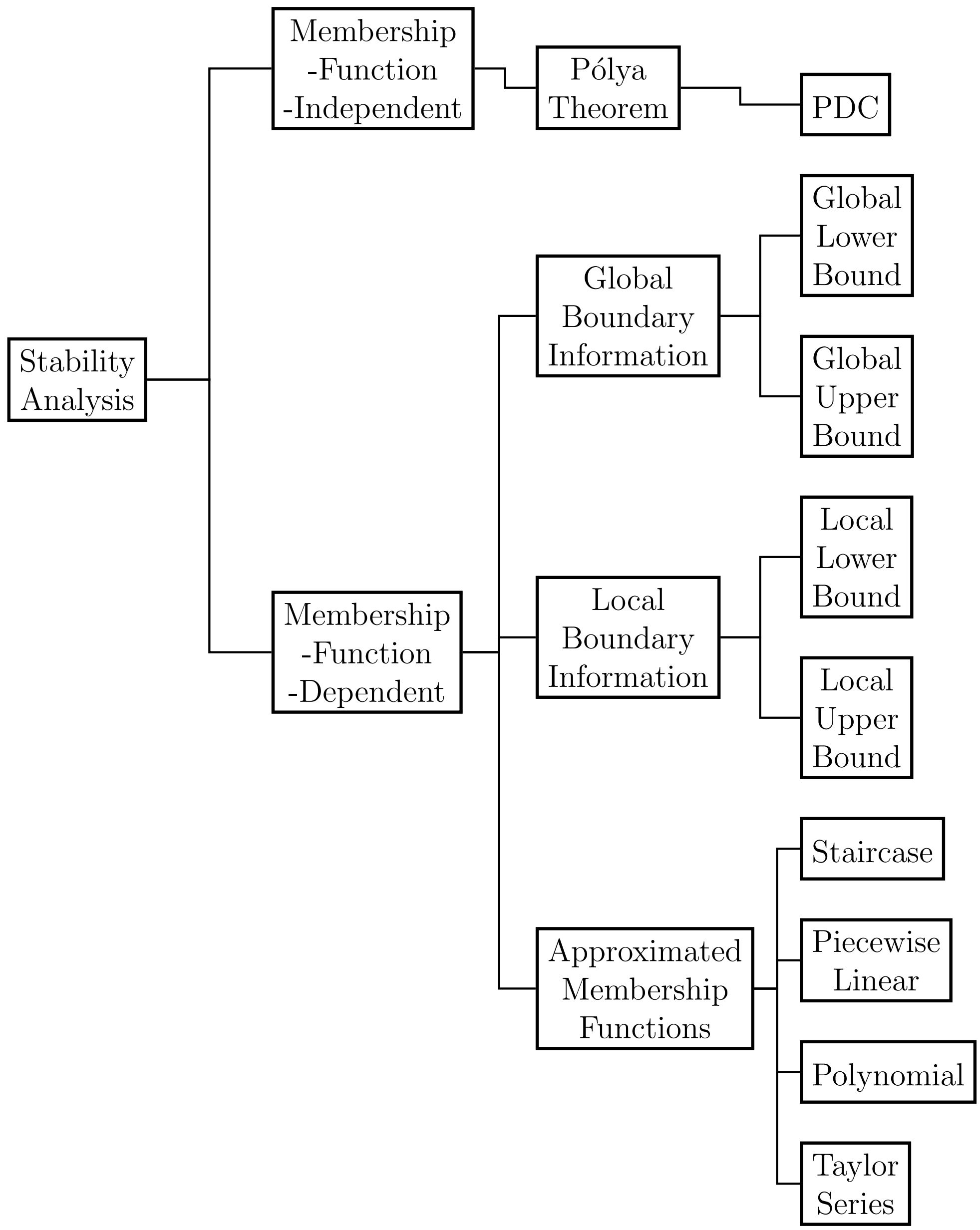

Figure 11: Techniques of stability analysis.

into it, $\dot{V}(\mathbf{x}(t))$ can be expressed in the following compact form.

$$\dot{V}(\mathbf{x}(t)) = \mathbf{z}(\mathbf{x}(t))^T \Big(\sum_{i=1}^{p} \sum_{j=1}^{c} w_i m_j \mathbf{Q}_{ij}(\mathbf{x}(t)) \Big) \mathbf{z}(\mathbf{x}(t)) \tag{14}$$

where $\mathbf{Q}_{ij}(\mathbf{x}(t))$ is a square matrix related to the system matrix $\mathbf{A}_i(\mathbf{x}(t))$, input matrix $\mathbf{B}_i(\mathbf{x}(t))$, feedback gain $\mathbf{G}_j(\mathbf{x}(t))$ and the Lyapunov function matrix $\mathbf{P}$, $\mathbf{z}(\mathbf{x}(t))$ is a row vector in $\mathbf{x}(t)$ satisfying $\mathbf{z}(\mathbf{0}) = \mathbf{0}$. Their dimensions are omitted and assumed to be compatible. Readers should refer to [11] and the references therein for the details of obtaining $\mathbf{Q}_{ij}(\mathbf{x}(t))$.

From (14), as w_i and m_j are of positive, $\dot{V}(\mathbf{x}(t)) \leq 0$ (equality holds for $\mathbf{z}(\mathbf{x}(t)) = \mathbf{0}$) can be achieved if $\mathbf{Q}_{ij}(\mathbf{x}(t)) < 0$ for all i and j. Consequently, by satisfying the stability conditions $\mathbf{P} > 0$ and $\mathbf{Q}_{ij}(\mathbf{x}(t)) < 0$, the PFMB control system (9) under the category of imperfectly matched premises is asymptotically stable in the sense of Lyapunov. These stability conditions are MFI as they do not depend on the membership functions w_i and m_j as they are dropped during the stability analysis.

MFI Stability Analysis using Pólya Theorem.
In the literature, under a special case that the fuzzy controller shares the same premise membership functions and number of rules as those of the fuzzy model, i.e., $\{m_1, \cdots, m_c\} = \{w_1, \cdots, w_p\}$ and $c = p$, which is in the category of perfectly matched premises as shown in Fig. 7 or known as PDC, the stability analysis results can be progressively relaxed using Pólya Theorem by considering the permutations of membership functions [46, 47, 50, 51, 52, 53, 54, 55].

Choosing $\{m_1, \cdots, m_c\} = \{w_1, \cdots, w_p\}$ and $c = p$, expanding (14), we have

$$\begin{aligned}
\dot{V}(\mathbf{x}(t)) &= \mathbf{z}(\mathbf{x}(t))^T \Big(\sum_{i=1}^{p} \sum_{j=1}^{p} w_i w_j \mathbf{Q}_{ij}(\mathbf{x}(t)) \Big) \mathbf{z}(\mathbf{x}(t)) \\
&= \mathbf{z}(\mathbf{x}(t))^T \Big(w_1 w_1 \mathbf{Q}_{11}(\mathbf{x}(t)) + w_1 w_2 \mathbf{Q}_{12}(\mathbf{x}(t)) \\
&\quad + \cdots + w_2 w_1 \mathbf{Q}_{21}(\mathbf{x}(t)) + \cdots + w_p w_p \mathbf{Q}_{pp}(\mathbf{x}(t)) \Big) \mathbf{z}(\mathbf{x}(t)) \\
&= \mathbf{z}(\mathbf{x}(t))^T \Big(w_1 w_1 \mathbf{Q}_{11}(\mathbf{x}(t)) + w_1 w_2 \big(\mathbf{Q}_{12}(\mathbf{x}(t)) + \mathbf{Q}_{21}(\mathbf{x}(t)) \big) \\
&\quad + \cdots + w_p w_p \mathbf{Q}_{pp}(\mathbf{x}(t)) \Big) \mathbf{z}(\mathbf{x}(t)).
\end{aligned} \tag{15}$$

It can be seen from (15) that some $\mathbf{Q}_{ij}(\mathbf{x}(t))$ terms share the same product term of membership functions [46, 47], for example, $\mathbf{Q}_{12}(\mathbf{x}(t))$ and $\mathbf{Q}_{21}(\mathbf{x}(t))$ share the same $w_1 w_2$. Consequently, $\dot{V}(\mathbf{x}(t)) < 0$ can be achieved if all grouped terms with the same product term of membership functions are negative definite such as $\mathbf{Q}_{12}(\mathbf{x}(t)) + \mathbf{Q}_{21}(\mathbf{x}(t)) < 0$, which relax the stability analysis conditions as it does not require every single $\mathbf{Q}_{ij}(\mathbf{x}(t)) < 0$ to be satisfied for all i and j as in the category of imperfectly matched premises.

In (15), the order of fuzzy summations is 2 as the product term of membership functions involves two membership functions, e.g., $w_i w_j$. In the following, multiplying $\sum_{k=1}^{p} w_k$ which equals 1 due to the property of membership functions in (3), to the left hand side of (15) gives below:

$$\begin{aligned}
\dot{V}(\mathbf{x}(t)) &= \mathbf{z}(\mathbf{x}(t))^T \Big(\sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{k=1}^{p} w_i w_j w_k \mathbf{Q}_{ij}(\mathbf{x}(t)) \Big) \mathbf{z}(\mathbf{x}(t)) \\
&= \mathbf{z}(\mathbf{x}(t))^T \Big(w_1 w_1 w_1 \mathbf{Q}_{11}(\mathbf{x}(t)) + \cdots + w_1 w_2 w_3 \mathbf{Q}_{12}(\mathbf{x}(t)) \\
&\quad + w_1 w_3 w_2 \mathbf{Q}_{13}(\mathbf{x}(t)) + \cdots + w_2 w_1 w_3 \mathbf{Q}_{21}(\mathbf{x}(t)) \\
&\quad + w_2 w_3 w_1 \mathbf{Q}_{23}(\mathbf{x}(t)) + \cdots + w_3 w_1 w_2 \mathbf{Q}_{31}(\mathbf{x}(t)) \\
&\quad + \cdots + w_3 w_2 w_1 \mathbf{Q}_{32}(\mathbf{x}(t)) + \cdots + w_p w_p w_p \mathbf{Q}_{pp}(\mathbf{x}(t)) \Big) \mathbf{z}(\mathbf{x}(t)) \\
&= \mathbf{z}(\mathbf{x}(t))^T \Big(w_1 w_1 w_1 \mathbf{Q}_{11}(\mathbf{x}(t)) + \cdots + w_1 w_2 w_3 \big(\mathbf{Q}_{12}(\mathbf{x}(t)) + \\
&\quad + \mathbf{Q}_{13}(\mathbf{x}(t)) + \mathbf{Q}_{21}(\mathbf{x}(t)) + \mathbf{Q}_{23}(\mathbf{x}(t)) + \mathbf{Q}_{31}(\mathbf{x}(t)) + \mathbf{Q}_{32}(\mathbf{x}(t)) \big) \\
&\quad + \cdots + w_p w_p w_p \mathbf{Q}_{pp}(\mathbf{x}(t)) \Big) \mathbf{z}(\mathbf{x}(t)).
\end{aligned} \tag{16}$$

Similar to the case of fuzzy summations of order 2, the $\mathbf{Q}_{ij}(\mathbf{x}(t))$ terms with the same product term of membership functions in (16), e.g., $w_1 w_2 w_3$, are grouped. For this case of fuzzy summations of order 3, six $\mathbf{Q}_{ij}(\mathbf{x}(t))$ terms can be grouped, e.g., $\mathbf{Q}_{12}(\mathbf{x}(t)) + \mathbf{Q}_{13}(\mathbf{x}(t)) + \mathbf{Q}_{21}(\mathbf{x}(t)) + \mathbf{Q}_{23}(\mathbf{x}(t)) + \mathbf{Q}_{31}(\mathbf{x}(t)) + \mathbf{Q}_{32}(\mathbf{x}(t))$, instead of two $\mathbf{Q}_{ij}(\mathbf{x}(t))$ terms in the case of fuzzy summations of order 2. Consequently, $\dot{V}(\mathbf{x}(t)) < 0$ can be achieved if all grouped terms with the same product terms of membership functions are negative definite such as $\mathbf{Q}_{12}(\mathbf{x}(t)) + \mathbf{Q}_{13}(\mathbf{x}(t)) + \mathbf{Q}_{21}(\mathbf{x}(t)) + \mathbf{Q}_{23}(\mathbf{x}(t)) + \mathbf{Q}_{31}(\mathbf{x}(t)) + \mathbf{Q}_{32}(\mathbf{x}(t)) < 0$. The sum of more $\mathbf{Q}_{ij}(\mathbf{x}(t))$ terms required to be negative implies that the stability conditions are more relaxed.

Along the same line of logic, in order to get more $\mathbf{Q}_{ij}(\mathbf{x}(t))$ terms to be grouped, we can increase the order of the fuzzy summations. By multiplying

$\sum_{i_1=1}^{p} \cdots \sum_{i_d=1}^{p} w_{i_1} \cdots w_{i_d}$ which equals 1 to the right hand side of (15), we can increase the order of fuzzy summations to $d+2$ shown as follows:

$$\dot{V}(\mathbf{x}(t)) = \mathbf{z}(\mathbf{x}(t))^T \Big(\sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{i_1=1}^{p} \cdots \sum_{i_d=1}^{p} w_i w_j w_{i_1} \cdots w_{i_d} \mathbf{Q}_{ij}(\mathbf{x}(t)) \Big) \mathbf{z}(\mathbf{x}(t)). \tag{17}$$

In this case, it is possible to group more $\mathbf{Q}_{ij}(\mathbf{x}(t))$ terms sharing the same product term of membership functions $w_i w_j w_{i_1} \cdots w_{i_d}$ for relaxing the stability conditions. The details can be found in [55].

Remark 7. *It should be noted that the above discussion is just to illustrate the concept of considering the permutations of membership functions for the relaxation of stability analysis results but some details are omitted. Readers should refer to the corresponding papers [46, 47, 50, 51, 52, 53, 54, 55] for the working details of stability analysis and the complete stability conditions.*

4.4.2. MFD Stability Analysis Techniques

The membership functions w_i and m_j contain a lot information of the nonlinear plant represented by the FMB/PFMB fuzzy model. By dropping them in the stability analysis, the stability conditions will be conservative as a lot of information is not considered. Furthermore, by satisfying the MFI stability conditions given in Section 4.4.1, the PFMB control system (9) with any form of membership functions w_i and m_j are guaranteed to be asymptotically stable. It also explains why the MFI stability conditions are conservative as the MFI stability analysis is for a family of PFMB control systems but not the one considered on hand. This observation gives a hint on relaxing the stability conditions, which can be achieved by bringing the information of membership functions into the stability conditions or equivalently reducing the size of the family of PFMB control systems to be considered, and motivates the MFD stability analysis.

Referring to Fig. 11, the MFD stability analysis can carry the global boundary information, local boundary information or approximated membership functions into the stability analysis, where details are given below.

Global Boundary Information.
Referring to (11) and denoting $h_{ij}(\mathbf{x}(t)) \triangleq w_i(\mathbf{x}(t))m_j(\mathbf{x}(t))$, there exists the global lower and upper bounds of the product term of membership function

$h_{ij}(\mathbf{x}(t))$. For example, assuming that a membership function $h_{ij}(x_1(t))$ depending on a single system state $x_1(t)$, which can be plotted as a figure as shown in Fig. 12. For the membership function $h_{ij}(x_1(t))$, its constant global lower and upper bounds are denoted as $\underline{\gamma}_{ij}$ and $\overline{\gamma}_{ij}$, respectively, satisfying $0 \leq \underline{\gamma}_{ij} \leq h_{ij}(x_1(t)) \leq \overline{\gamma}_{ij} \leq 1$. It should be noted that $h_{ij}(x_1(t))$ becomes $h_{ij}(\mathbf{x}(t))$ for general cases in the following analysis.

The global boundary information of membership functions will be brought into the stability analysis through slack polynomial matrices, $\underline{\mathbf{R}}_{ij}(\mathbf{x}(t)) \geq 0$ carrying the global lower boundary information and $\overline{\mathbf{R}}_{ij}(\mathbf{x}(t)) \geq 0$ carrying the global upper boundary information, which satisfy

$$\sum_{i=1}^{p}\sum_{j=1}^{c}\big(h_{ij}(\mathbf{x}(t)) - \underline{\gamma}_{ij}\big)\underline{\mathbf{R}}_{ij}(\mathbf{x}(t)) \geq 0, \tag{18}$$

$$\sum_{i=1}^{p}\sum_{j=1}^{c}\big(\overline{\gamma}_{ij} - h_{ij}(\mathbf{x}(t))\big)\overline{\mathbf{R}}_{ij}(\mathbf{x}(t)) \geq 0. \tag{19}$$

Consider the category of imperfectly matched premises. Adding (18) and (19) to (14), we have

$$\begin{aligned}
\dot{V}(\mathbf{x}(t)) &= \mathbf{z}(\mathbf{x}(t))^T\Big(\sum_{i=1}^{p}\sum_{j=1}^{c}h_{ij}(\mathbf{x}(t))\mathbf{Q}_{ij}(\mathbf{x}(t))\Big)\mathbf{z}(\mathbf{x}(t)) \\
&\leq \mathbf{z}(\mathbf{x}(t))^T\Big(\sum_{i=1}^{p}\sum_{j=1}^{c}h_{ij}(\mathbf{x}(t))\mathbf{Q}_{ij}(\mathbf{x}(t)) + \sum_{i=1}^{p}\sum_{j=1}^{c}\big(h_{ij}(\mathbf{x}(t)) - \underline{\gamma}_{ij}\big)\underline{\mathbf{R}}_{ij}(\mathbf{x}(t)) \\
&\quad + \sum_{i=1}^{p}\sum_{j=1}^{c}\big(\overline{\gamma}_{ij} - h_{ij}(\mathbf{x}(t))\big)\overline{\mathbf{R}}_{ij}(\mathbf{x}(t))\Big)\mathbf{z}(\mathbf{x}(t)) \\
&= \mathbf{z}(\mathbf{x}(t))^T\Big(\sum_{i=1}^{p}\sum_{j=1}^{c}h_{ij}(\mathbf{x}(t))\big(\mathbf{Q}_{ij}(\mathbf{x}(t)) + \underline{\mathbf{R}}_{ij}(\mathbf{x}) - \overline{\mathbf{R}}_{ij}(\mathbf{x}(t))\big) \\
&\quad - \sum_{r=1}^{p}\sum_{s=1}^{c}\underline{\gamma}_{rs}\underline{\mathbf{R}}_{rs}(\mathbf{x}(t)) + \sum_{r=1}^{p}\sum_{s=1}^{c}\overline{\gamma}_{rs}\overline{\mathbf{R}}_{rs}(\mathbf{x}(t))\Big)\mathbf{z}(\mathbf{x}(t)).
\end{aligned} \tag{20}$$

It can be seen from (20) that $\dot{V}(\mathbf{x}(t)) \leq 0$ (equality holds for $\mathbf{z}(\mathbf{x}(t)) = \mathbf{0}$)

can be achieved by satisfying

$$\begin{aligned}&\mathbf{Q}_{ij}(\mathbf{x}(t)) + \underline{\mathbf{R}}_{ij}(\mathbf{x}) - \overline{\mathbf{R}}_{ij}(\mathbf{x}(t)) \\ &\quad - \sum_{r=1}^{p}\sum_{s=1}^{c} \underline{\gamma}_{rs}\underline{\mathbf{R}}_{rs}(\mathbf{x}(t)) + \sum_{r=1}^{p}\sum_{s=1}^{c} \overline{\gamma}_{rs}\overline{\mathbf{R}}_{rs}(\mathbf{x}(t)) < 0, \forall\, i, j. \qquad (21)\end{aligned}$$

Comparing with the stability conditions for the category of imperfectly matched premises under MFI stability analysis in Section 4.4.1 which require $\mathbf{Q}_{ij}(\mathbf{x}(t)) < 0$ for all i and j, the global lower and upper boundary information, $\underline{\gamma}_{ij}$ and $\overline{\gamma}_{ij}$, are included in the stability conditions (21), which are able to relax the stability analysis results.

When one stability condition is said to be more relaxed than others, it is equivalent to that the other stability conditions are the subset of the relaxed one. Denoting $\mathbf{R}_{ij}(\mathbf{x}(t)) \triangleq \underline{\mathbf{R}}_{ij}(\mathbf{x}(t)) - \overline{\mathbf{R}}_{ij}(\mathbf{x}(t)) - \sum_{r=1}^{p}\sum_{s=1}^{c}\underline{\gamma}_{rs}\underline{\mathbf{R}}_{rs}(\mathbf{x}(t)) + \sum_{r=1}^{p}\sum_{s=1}^{c}\overline{\gamma}_{rs}\overline{\mathbf{R}}_{rs}(\mathbf{x}(t))$, the stability condition (21) can be expressed in a compact form: $\mathbf{Q}_{ij}(\mathbf{x}(t)) + \mathbf{R}_{ij}(\mathbf{x}(t)) < 0$. There are two cases to consider, namely $\mathbf{R}_{ij}(\mathbf{x}(t)) < 0$ and $\mathbf{R}_{ij}(\mathbf{x}(t)) \geq 0$. When there exist some $\underline{\mathbf{R}}_{ij}(\mathbf{x}(t))$ and $\overline{\mathbf{R}}_{ij}(\mathbf{x}(t))$ such that $\mathbf{R}_{ij}(\mathbf{x}(t)) < 0$, $\mathbf{Q}_{ij}(\mathbf{x}(t)) + \mathbf{R}_{ij}(\mathbf{x}(t)) < 0$ would be easier to be satisfied than $\mathbf{Q}_{ij}(\mathbf{x}(t)) < 0$. When some $\underline{\mathbf{R}}_{ij}(\mathbf{x}(t))$ and $\overline{\mathbf{R}}_{ij}(\mathbf{x}(t))$ cannot be found to have $\mathbf{R}_{ij}(\mathbf{x}(t)) < 0$, choosing $\underline{\mathbf{R}}_{ij}(\mathbf{x}(t)) = \overline{\mathbf{R}}_{ij}(\mathbf{x}(t)) = \mathbf{0}$ will reduce the stability conditions $\mathbf{Q}_{ij}(\mathbf{x}(t)) + \mathbf{R}_{ij}(\mathbf{x}(t)) < 0$ to $\mathbf{Q}_{ij}(\mathbf{x}(t)) < 0$. As a result, it can be concluded that the stability condition (21) with global boundary information are more relaxed compared with the ones without boundary information or at least equivalent.

Remark 8. *When the category of perfectly matched premises is considered in* (20)*, the techniques using Pólya theorem in Section 4.4.1 can also be employed to further relax the MFD stability conditions.*

Remark 9. *The category of partially matched premises is a compromise between perfectly and imperfectly matched premises. Choosing $c = p$ in* (14)*, it can be written as*

$$\begin{aligned}\dot{V}(\mathbf{x}(t)) = \mathbf{z}(\mathbf{x}(t))^T\Big(&\sum_{i=1}^{p}\sum_{j=1}^{p} w_i(\mathbf{x}(t))w_j(\mathbf{x}(t))\mathbf{Q}_{ij}(\mathbf{x}(t)) \\ &+ \sum_{i=1}^{p}\sum_{j=1}^{p}(h_{ij}(\mathbf{x}(t)) - w_i(\mathbf{x}(t))w_j(\mathbf{x}(t)))\mathbf{Q}_{ij}(\mathbf{x}(t))\Big)\mathbf{z}(\mathbf{x}(t))\end{aligned}$$

$$
\begin{aligned}
&= \mathbf{z}(\mathbf{x}(t))^T \Big(\sum_{i=1}^{p} \sum_{j=1}^{p} (w_i(\mathbf{x}(t)) w_j(\mathbf{x}(t)) + \underline{\sigma}_{ij}) \mathbf{Q}_{ij}(\mathbf{x}(t)) \\
&\quad + \sum_{i=1}^{p} \sum_{j=1}^{p} (h_{ij}(\mathbf{x}(t)) - w_i(\mathbf{x}(t)) w_j(\mathbf{x}(t)) - \underline{\sigma}_{ij}) \mathbf{Q}_{ij}(\mathbf{x}(t)) \Big) \mathbf{z}(\mathbf{x}(t)) \\
&\leq \mathbf{z}(\mathbf{x}(t))^T \Big(\sum_{i=1}^{p} \sum_{j=1}^{p} (w_i(\mathbf{x}(t)) w_j(\mathbf{x}(t)) + \underline{\sigma}_{ij}) \mathbf{Q}_{ij}(\mathbf{x}(t)) \\
&\quad + \sum_{i=1}^{p} \sum_{j=1}^{p} (\overline{\sigma}_{ij} - \underline{\sigma}_{ij}) \mathbf{Y}_{ij}(\mathbf{x}(t)) \Big) \mathbf{z}(\mathbf{x}(t)) \\
&= \mathbf{z}(\mathbf{x}(t))^T \sum_{i=1}^{p} \sum_{j=1}^{p} w_i(\mathbf{x}(t)) w_j(\mathbf{x}(t)) \Big(\mathbf{Q}_{ij}(\mathbf{x}(t)) \\
&\quad + \sum_{r=1}^{p} \sum_{s=1}^{p} \big(\underline{\sigma}_{rs} \mathbf{Q}_{rs}(\mathbf{x}(t)) + (\overline{\sigma}_{rs} - \underline{\sigma}_{rs}) \mathbf{Y}_{rs}(\mathbf{x}(t)) \big) \Big) \mathbf{z}(\mathbf{x}(t)) \\
&= \mathbf{z}(\mathbf{x}(t))^T \Big(\sum_{i=1}^{p} \sum_{j=1}^{p} w_i(\mathbf{x}(t)) w_j(\mathbf{x}(t)) \mathbf{H}_{ij}(\mathbf{x}(t)) \Big) \mathbf{z}(\mathbf{x}(t)) \qquad (22)
\end{aligned}
$$

where $\underline{\sigma}_{ij}$ *and* $\overline{\sigma}_{ij}$ *are constants satisfying* $\underline{\sigma}_{ij} \leq h_{ij}(\mathbf{x}(t)) - w_i(\mathbf{x}(t)) w_j(\mathbf{x}(t)) \leq \overline{\sigma}_{ij}$, $\mathbf{Y}_{ij}(\mathbf{x}(t)) \geq 0$, $\mathbf{Y}_{ij}(\mathbf{x}(t)) \geq \mathbf{Q}_{ij}(\mathbf{x}(t))$ *and* $\mathbf{H}_{ij}(\mathbf{x}(t)) = \mathbf{Q}_{ij}(\mathbf{x}(t)) + \sum_{r=1}^{p} \sum_{s=1}^{p} (\underline{\sigma}_{rs} \mathbf{Q}_{rs}(\mathbf{x}(t)) + (\overline{\sigma}_{rs} - \underline{\sigma}_{rs}) \mathbf{Y}_{rs}(\mathbf{x}(t)))$.

It can be seen that the form in (22) is under the category of perfectly matched premises, which means that the stability analysis technique using Pólya theorem can be applied to obtain and relax the stability conditions. As the category of partially matched premises can be represented by the category of perfectly matched premises, all the MFI and MFD techniques developed for the category of perfectly matched premises can be applied.

Regional Boundary Information.
Although the global boundary information can provide some useful information about the membership functions, it can be seen from Fig. 12 that the global lower and upper bounds contain very limited information. For example, the position and the shape of membership functions can be changed but the global boundary information might be the same. It gives a clue that the position and shape information of the membership functions should be included for the relaxation of stability analysis results.

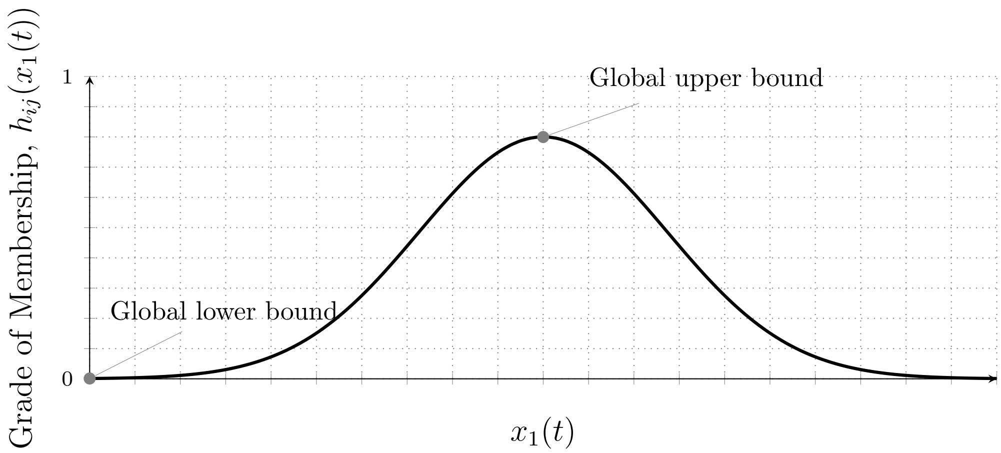

Figure 12: Global boundary information of a membership function.

By dividing the operating domain into connected sub-domains as shown in Fig. 13, the local lower and upper bounds can be obtained, which capture the position and shape information of the membership functions. When the position and/or the shape of membership functions change(s), the local lower and upper bounds might change and the change is more sensitive than the global boundary information. In Fig. 13, the number of sub-domains is 3, which can be any number chosen by control engineers. When more sub-domains are considered, more information of the membership functions will be brought into the stability analysis resulting in more relaxed stability conditions.

The stability analysis is the same as that in the case of global boundary information where the only difference is that the local boundary information and local slack matrices are used. Assuming that the operating domain Φ is divided into D local connected sub-domains Φ_k, i.e., $\Phi = \cup_{k=1}^{D} \Phi_k$, its constant local lower and upper bounds for the k-th local sub-domain are denoted as $\underline{\gamma}_{ijk}$ and $\overline{\gamma}_{ijk}$, respectively, satisfying $0 \leq \underline{\gamma}_{ijk} \leq h_{ij}(\mathbf{x}(t)) \leq \overline{\gamma}_{ijk} \leq 1$, $k = 1$, $\ldots$, D. By replacing the global lower and upper bounds $\underline{\gamma}_{ij}$ and $\overline{\gamma}_{ij}$, and the slack matrices $\underline{\mathbf{R}}_{ij}$ and $\overline{\mathbf{R}}_{ij}$ by $\underline{\gamma}_{ijk}$, $\overline{\gamma}_{ijk}$, $\underline{\mathbf{R}}_{ijk}$ and $\overline{\mathbf{R}}_{ijk}$, respectively, in (18) to (20), we can obtain $\dot{V}(\mathbf{x}(t))$ as follows:

$$\begin{aligned}\dot{V}(\mathbf{x}(t)) \leq \mathbf{z}(\mathbf{x}(t))^T \sum_{k=1}^{D} \frac{\xi_k(\mathbf{x}(t))}{\sum_{l=1}^{D} \xi_l(\mathbf{x}(t))} \\ \times \Big(\sum_{i=1}^{p} \sum_{j=1}^{c} h_{ij}(\mathbf{x}(t)) \big(\mathbf{Q}_{ij}(\mathbf{x}(t)) + \underline{\mathbf{R}}_{ijk}(\mathbf{x}) - \overline{\mathbf{R}}_{ijk}(\mathbf{x}(t)) \\ - \sum_{r=1}^{p} \sum_{s=1}^{c} \underline{\gamma}_{rsk} \underline{\mathbf{R}}_{rsk}(\mathbf{x}(t)) + \sum_{r=1}^{p} \sum_{s=1}^{c} \overline{\gamma}_{rsk} \overline{\mathbf{R}}_{rsk}(\mathbf{x}(t)) \big) \Big) \mathbf{z}(\mathbf{x}(t)), \\ \forall\, \mathbf{x}(t) \in \Phi_k; k = 1, \ldots, D. \qquad (23)\end{aligned}$$

where $\xi_k(\mathbf{x}(t)) = \begin{cases} 1, & \mathbf{x}(t) \in \Phi_k \\ 0, & \text{otherwise} \end{cases}$.

It can be seen from (23) that $\dot{V}(\mathbf{x}(t)) \leq 0$ (equality holds for $\mathbf{z}(\mathbf{x}(t)) = \mathbf{0}$) can be achieved by satisfying

$$\begin{aligned}\mathbf{Q}_{ij}(\mathbf{x}(t)) + \underline{\mathbf{R}}_{ijk}(\mathbf{x}) - \overline{\mathbf{R}}_{ijk}(\mathbf{x}(t)) \\ - \sum_{r=1}^{p} \sum_{s=1}^{c} \underline{\gamma}_{rs} \underline{\mathbf{R}}_{rsk}(\mathbf{x}(t)) + \sum_{r=1}^{p} \sum_{s=1}^{c} \overline{\gamma}_{rs} \overline{\mathbf{R}}_{rsk}(\mathbf{x}(t)) < 0, \\ \forall\, \mathbf{x}(t) \in \Phi_k; k = 1, \ldots, D. \qquad (24)\end{aligned}$$

When $D = 1$, the stability condition (24) is reduced to that in (21), i.e., the case of global boundary information. Keep increasing D will take more and more local lower and upper boundary information into the stability conditions that the conservativeness will be progressively reduced. However, the number of stability conditions will increase, which increases the computational demand on finding a feasible solution (if any) to the stability conditions.

Approximated Membership Functions.
The stability analysis can be facilitated under certain types of membership functions such as the staircase membership functions, piecewise linear membership functions and polynomial membership functions [10, 11] as shown in Fig. 14, which considers these membership functions depending on $x_1(t)$ only for simplicity but the concept can be extended to hyper-dimensional space.

Referring to (11), we denote those just mentioned particular type of membership functions as $\hat{h}_{ij}(\mathbf{x}(t))$. It is assumed that the nonlinear plant can be

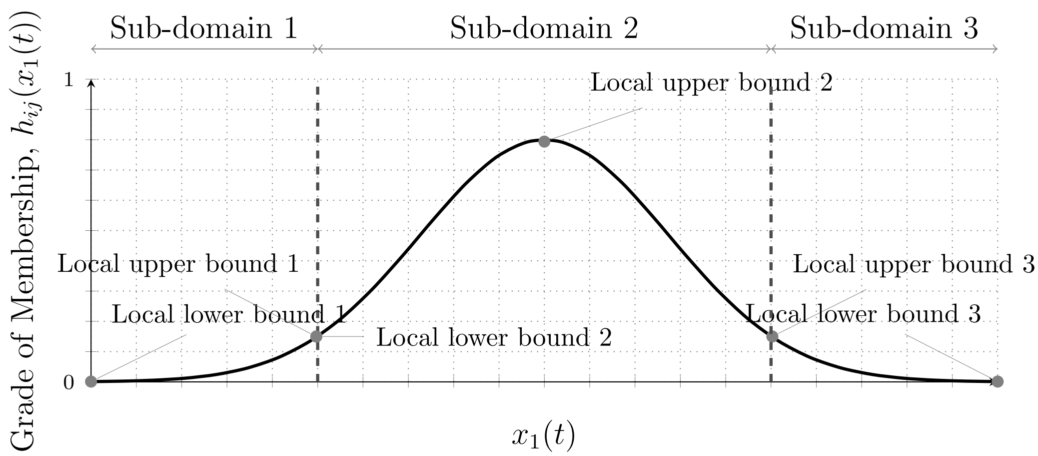

Figure 13: Local boundary information of a membership function.

represented by a T-S/polynomial fuzzy model with one of these particular type of membership functions. Then, we have $\dot{V}(\mathbf{x}(t))$ as follows:

$$\dot{V}(\mathbf{x}(t)) = \mathbf{z}(\mathbf{x}(t))^T \Big(\sum_{i=1}^{p} \sum_{j=1}^{c} \hat{h}_{ij}(\mathbf{x}(t)) \mathbf{Q}_{ij}(\mathbf{x}(t)) \Big) \mathbf{z}(\mathbf{x}(t)). \tag{25}$$

To make sure that $\dot{V}(\mathbf{x}(t)) \leq 0$ (equality holds when $\mathbf{x}(t) = \mathbf{0}$), from (25), we need to satisfy the following condition:

$$\sum_{i=1}^{p} \sum_{j=1}^{c} \hat{h}_{ij}(\mathbf{x}(t)) \mathbf{Q}_{ij}(\mathbf{x}(t)) < 0. \tag{26}$$

Referring to Fig. 14, if $\hat{h}_{ij}(\mathbf{x}(t))$ is a staircase membership function, it can be seen that $\hat{h}_{ij}(\mathbf{x}(t))$ takes some discrete values of membership grades for different values of $\mathbf{x}(t)$ as the value of $\hat{h}_{ij}(\mathbf{x}(t))$ does not change for certain range of $\mathbf{x}(t)$. As a result, if we can make sure that the stability condition (26) is satisfied for all discrete values $\hat{h}_{ij}(\mathbf{x}(t))$ for all $\mathbf{x}(t)$, $\dot{V}(\mathbf{x}(t)) < 0$ can be achieved. The detailed analysis can be found in [136, 11].

When $\hat{h}_{ij}(\mathbf{x}(t))$ is a piecewise linear membership function, referring to the dotted line in Fig. 14, it can be characterized by some sample points denoted as $c_{ij,1}$ to $c_{ij,5}$. For each segment, for example, any value in between $c_{ij,1}$ and

$c_{ij,2}$ can be represented by these two end points through interpolation. As a result, the stability of the PFMB control system is guaranteed if it is stable at all these sample points. For example, considering 5 sample points as in Fig. 14, $\dot{V}(\mathbf{x}(t)) < 0$ can be achieved if $\sum_{i=1}^{p}\sum_{j=1}^{c} c_{ij,k}\mathbf{Q}_{ij}(\mathbf{x}(t)) < 0$ for k = 1, 2, 3, 4, 5. The detailed analysis and extension to hyper-dimensional membership functions can be found in [76, 11].

When $\hat{h}_{ij}(\mathbf{x}(t))$ takes the form of polynomial function, which can be handled by SOS approach, the concept remains more or less the same. Referring to the dashed-dotted line in Fig. 14 as an example, $\hat{h}_{ij}(x_1(t))$ (the membership functions in the figure depends on x_1 only) is a piecewise polynomial function, i.e., $\hat{h}_{ij}(x_1(t))$ can be a different polynomial function in different segment. In this figure, we have 4 segments (the k-th segment is in between $c_{ij,k}$ and $c_{ij,k+1}$, k = 1, 2, 3, 4) and denote $\hat{h}_{ij}(x_1(t))$ as a polynomial function $p_{ij,k}(\mathbf{x}(t))$ in the k-th segment. $\dot{V}(\mathbf{x}(t)) < 0$ can be achieved if $\sum_{i=1}^{p}\sum_{j=1}^{c} p_{ij,k}(\mathbf{x}(t))\mathbf{Q}_{ij}(\mathbf{x}(t)) < 0$ for k = 1, 2, 3, 4. The detailed analysis and extension to hyper-dimensional membership functions can be found in [85, 11].

In the above, we assume that the membership functions are one of the particular types. However, in general, the membership functions can take any form. In order to extend the idea to general membership functions, we can approximate the original membership functions $h_{ij}(\mathbf{x}(t))$ with the particular type of membership functions $\hat{h}_{ij}(\mathbf{x}(t))$ with the consideration of the approximation error.

From (11), we have $\dot{V}(\mathbf{x}(t))$ as follows:

$$
\begin{aligned}
\dot{V}(\mathbf{x}(t)) &= \mathbf{z}(\mathbf{x}(t))^T\Big(\sum_{i=1}^{p}\sum_{j=1}^{c} h_{ij}(\mathbf{x}(t))\mathbf{Q}_{ij}(\mathbf{x}(t))\Big)\mathbf{z}(\mathbf{x}(t)) \\
&= \mathbf{z}(\mathbf{x}(t))^T\Big(\sum_{i=1}^{p}\sum_{j=1}^{c} \big(\hat{h}_{ij}(\mathbf{x}(t)) + \Delta\underline{h}_{ij} && (27)\\
&\quad + h_{ij}(\mathbf{x}(t)) - \hat{h}_{ij}(\mathbf{x}(t)) - \Delta\underline{h}_{ij}\big)\mathbf{Q}_{ij}(\mathbf{x}(t))\Big)\mathbf{z}(\mathbf{x}(t)) \\
&\le \mathbf{z}(\mathbf{x}(t))^T\Big(\sum_{i=1}^{p}\sum_{j=1}^{c} \big((\hat{h}_{ij}(\mathbf{x}(t)) + \Delta\underline{h}_{ij})\mathbf{Q}_{ij}(\mathbf{x}(t)) && (28)\\
&\quad + (\Delta\overline{h}_{ij} - \Delta\underline{h}_{ij})\mathbf{Y}_{ij}(\mathbf{x}(t))\big)\Big)\mathbf{z}(\mathbf{x}(t))
\end{aligned}
$$

(29)

where the approximation errors $\Delta\underline{h}_{ij}$ and $\Delta\overline{h}_{ij}$ are constants satisfying $\Delta\underline{h}_{ij} \leq h_{ij}(\mathbf{x}(t)) - \hat{h}_{ij}(\mathbf{x}(t)) \leq \Delta\overline{h}_{ij}$ for all $\mathbf{x}(t)$ or the domain of interest; $\mathbf{Y}_{ij}(\mathbf{x}(t)) \geq 0$ and $\mathbf{Y}_{ij}(\mathbf{x}(t)) \geq \mathbf{Q}_{ij}(\mathbf{x}(t))$.

From (27), $\dot{V}(\mathbf{x}(t)) < 0$ can be achieved if $\sum_{i=1}^{p}\sum_{j=1}^{c}\big((\hat{h}_{ij}(\mathbf{x}(t)) + \Delta\underline{h}_{ij})\mathbf{Q}_{ij}(\mathbf{x}(t)) + (\Delta\overline{h}_{ij} - \Delta\underline{h}_{ij})\mathbf{Y}_{ij}(\mathbf{x}(t))\big) < 0$, where $\hat{h}_{ij}(\mathbf{x}(t))$ can be any particular membership functions mentioned above. It can be seen that this stability condition contains the approximated membership functions, which is thus MFD. The magnitude of the approximation error plays an important role to reduce the conservativeness of stability conditions. By employing more sample points to construct the approximated membership functions, the approximation error can be reduced, however, the number of stability conditions will be increased resulting in increasing computational demand on finding a feasible solution.

In fact, the staircase membership functions, piecewise linear membership functions and polynomial membership functions can be represented by Taylor series of orders 0, 1 and higher than or equal to 2, respectively. The above concept using approximated membership functions has been generalized using the Taylor series [85, 11].

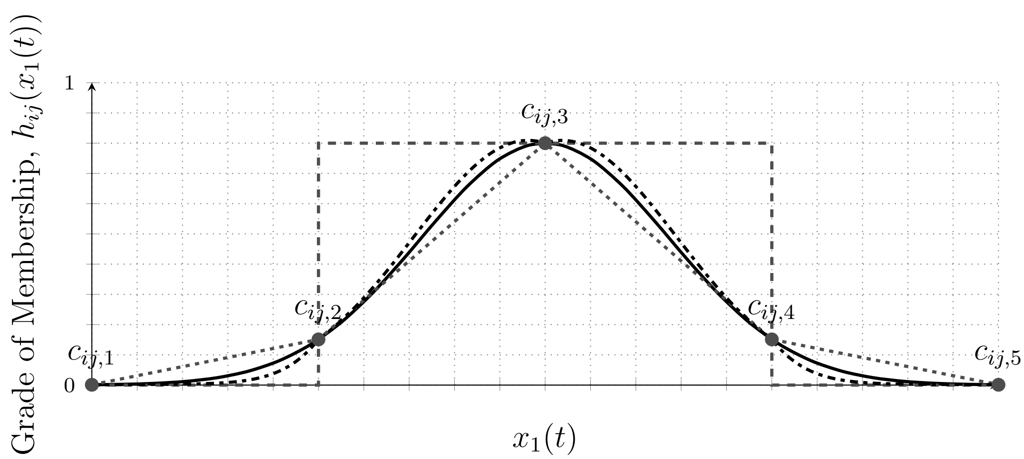

Figure 14: Membership functions. Solid line: original membership function. Dash line: staircase membership function. Dotted line: piecewise linear membership function. Dash-dotted line: Taylor series membership function.

Remark 10. *Referring to* (26)*, when the original membership functions are considered in the first place for stability analysis, the stability condi-*

tion becomes $\sum_{i=1}^{p}\sum_{j=1}^{c} h_{ij}(\mathbf{x}(t))\mathbf{Q}_{ij}(\mathbf{x}(t)) < 0$*, which is the most straightforward stability analysis results. However, when the membership function* $h_{ij}(\mathbf{x}(t))$ *is a general form,* $\sum_{i=1}^{p}\sum_{j=1}^{c} h_{ij}(\mathbf{x}(t))\mathbf{Q}_{ij}(\mathbf{x}(t)) < 0$ *contains an infinite number of stability conditions as it has to be satisfied by every single value of* $h_{ij}(\mathbf{x}(t))$*. When a particular type of membership functions mentioned above is used to approximate the original ones, the stability conditions* $\sum_{i=1}^{p}\sum_{j=1}^{c} h_{ij}(\mathbf{x}(t))\mathbf{Q}_{ij}(\mathbf{x}(t)) < 0$ *can be approximated by* (26) *with the consideration of approximation error, of which the number of stability conditions becomes finite due to the favorable property of the approximated membership functions. It is not practical to solve infinite number of stability conditions. When the number of stability conditions becomes finite, existing software package can be employed to solve a feasible solution (if any) numerically.*

5. Application Issues

In the above sections, they are all on the theoretical side and the main focus is on the development of stability conditions using MFD approach aiming to relaxing the conservativeness of stability analysis results. Bringing the developed theory to practice is always an ultimate goal but challenging due to many factors ranging from modeling accuracy, design concerns, assumptions to implementation issues.

Fig. 15, which is an adaptation of Fig. 3, shows the simple design cycle of bringing the theoretical design to practical implementation. In the figure, the blocks "Control Design" and " Stability Conditions" are mainly related to theoretical aspects while the blocks "System Modeling" and "Implementation" are mainly related the practical aspects.

When dealing with real applications using the FMB/PFMB control approach, there are three common fuzzy modeling approaches available in the literature:

1. Sector Nonlinearity Method [60, 61]: The mathematical model of the nonlinear plant can be exactly converted to T-S fuzzy model [60] or polynomial fuzzy model [61] locally or globally depending on the constraints considered such as the range of operating domain. This method can lead to some good modeling accuracy regarding to the nonlinear plant when the mathematical model is accurate enough. One example is DC-DC power converters [192, 193, 194], which is composed of mainly

electronic components. When mechanical systems are considered such as robot hands, wheeled mobile robots [195, 196] and bolt-tightening control problem [101], the moving parts of the systems, such as gear box, motors, are difficult to be modeled exactly such as the backlash and fraction. In some cases, to reduce the complexity of the (T-S or polynomial) fuzzy model, some complex terms can be simplified, for example, approximated by simple forms. Then, after replacing the complex terms by simple terms, sector nonlinearity method is applied to obtain a fuzzy model which is an approximation of the original nonlinear plant.

2. Fuzzy Aggregation of Local Linearized Models: A simplified (T-S or polynomial) fuzzy model may be more suitable to real applications from the analysis and implementation point of view. The idea is first by taking some sample operating points on the mathematical model. Corresponding to each sample operating point, a linearized model (local linear or polynomial model) is obtained. Then, some simple membership functions, such as S-, Z-, triangle and trapezoid membership functions, can be used to aggregate the linearized models to form a fuzzy model in the form of, e.g., (2). A successful example using this modeling technique can be found in [197], which deals with the tracking control problem for a prototype continuum manipulator. The PFMB controller demonstrated better tracking control performance than some traditional control methods.
3. System Identification Techniques [40, 41]: The mathematical model of the nonlinear plant may not be easy to obtain. By collecting the input-output data from experiment, system identification technique can be applied to construct the fuzzy model. It was proposed in [40, 41] that the structure and parameters of the matrices of the local models in the fuzzy model can be obtained by applying some system identification techniques to the collected input-output data. The rules of the fuzzy model can be determined based on the input-output data distribution. Membership functions are then employed to combine the local model to form the fuzzy model. A successful example using this idea can be found in the bolt-tightening control problem for wind turbine systems [101].

In general, employing the FMB control approach, it starts from the construction of fuzzy model where the modeling issues are briefly discussed above. A fuzzy controller is then considered for the control problem where the number of rules and membership functions are chosen for the initial design. Then, applying the stability conditions to obtain the feedback gains for the fuzzy controller. It is followed by implementing the fuzzy control strategy. Testing and evaluation are then performed to check if it is working. This is the forward path indicated by the gray arrows in Fig. 15. If it is not working well, revisiting of the processes in 'System Modeling", "Controller Design", "Stability Conditions" and "Implementation" is necessary, which is indicated by the feedback path in dotted line in Fig. 15.

Various reasons may cause the design given by the forward path not working. In the block "System Modeling", the accuracy of the fuzzy model plays a crucial role, which needs to be revised, for exampling, by factoring in the issues of uncertainties (such as parameter uncertainties), oversimplification, insufficient number of local models, disturbances, noise during data collection (e.g., a filter may be needed to collect clean data), measurement precision for data collection (e.g., more precise sensors may be needed), etc.

In the blocks "Controller Design" and "Implementation", the fuzzy controller will be physically implemented. All components contain uncertainties and nonlinearity, which may not be adequately considered in the stability analysis. In some cases, filters are employed in the input and output stages of the fuzzy controller, which may not be considered adequately as well in the stability analysis. Other practical issues such as saturation may cause the fuzzy control strategy not working well. Fine adjustment on the feedback gains obtained from solving the stability conditions may be required to make the fuzzy controller work. Otherwise, digital implementation of fuzzy controller, which is not discussed in this review, may cause other issues, such as sampling process, zero-order-hold process and quantization, making the fuzzy controller not work well.

In the block "Stability Conditions", various stability conditions are available in the literature, which demonstrate different levels of conservativeness and are subject to different assumptions and limitations. While some stability conditions may not offer feasible solution to the control problem on hand, other more relaxed stability conditions can be tried. Before applying the stability conditions, their assumptions and limitations have to be considered.

In this review paper, we only consider the stability analysis issue. Performance and robustness are other fundamental issues [135, 198, 199, 72, 92, 200, 201, 202], which should be considered in the design cycle.

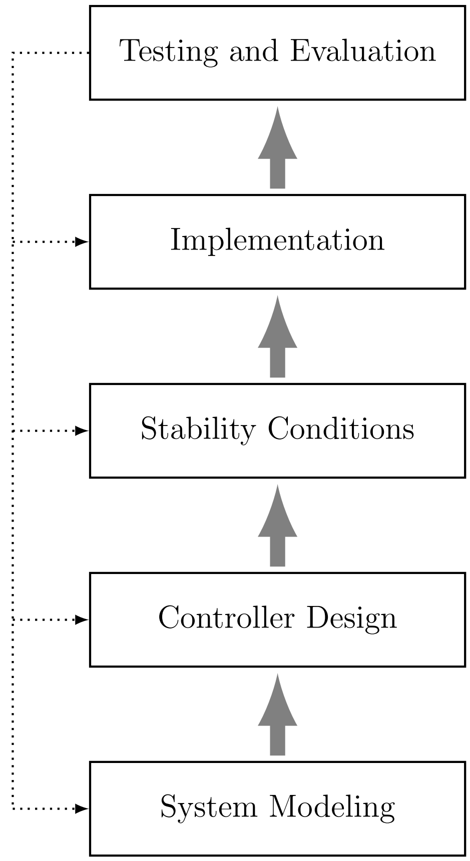

Figure 15: Design cycle of putting theory to practice.

6. Future Research

In the following, a brief discussion on future research of FMB/PFMB control systems from the context of MFD analysis is given below.

1. Stability Analysis using Lyapunov Functions: Referring to Fig. 6, the form of Lyapunov function plays an important role in relaxing the stability analysis results. Exploring a new form of Lyapunov function will remain as a research direction offering high impact of results on stability analysis.
2. Stability Analysis using MFD Techniques: Referring to Fig. 6, the author proposed to use local/global boundary of membership functions, and approximated membership functions for stability analysis. The

main idea is extract the information of membership functions and use it in the stability conditions. The ways of capturing the information of membership functions and the techniques of utilizing it in the stability analysis will point a direction for achieving more relaxed stability analysis results.

3. Performance/Robustness Analysis: The analyses of performance and robustness are comparatively less considered in the literature but they are essential issues for control applications. The MFD analysis and techniques discussed in this paper can be extended to performance/robustness analysis.
4. Control Systems and their Control Methodologies: The MFD analysis and techniques discussed in this paper are mainly for FMB/PFMB control systems with state-feedback fuzzy controller. It can be extended to those as shown, but not limited to, in Figs. 2 and 4.
5. IT2 FMB/PFMB Control Systems: In this paper, only type-1 FMB/PFMB control systems are discussed. The proposed MFD analysis and techniques can be applied to IT2 FMB/PFMB control systems where limited results have been received in the literature. Due to the existence of of uncertainties captured by IT2 fuzzy systems [99, 100, 101], the IT2 FMB/PFMB control systems are in the categories of partially/imperfectly matched premises, which can be handled by the proposed MFD analysis and techniques.
6. Applications: Bringing the theory to applications can demonstrate that the analysis results can be applied practically. In the future, this can be a direction in promoting the MFD analysis techniques and showing that it can be an effective tool for achieving stabilizable fuzzy controllers.

7. Conclusion

This paper presents an overview on the development of the stability analysis of continuous-time FMB/PFMB control systems, with emphasis on state-feedback control techniques, by reviewing the milestones and identifying the important stages happened in the past few decades. It is followed by discussing the stability analysis of continuous-time FMB/PFMB control systems over four aspects, namely the types of membership-function matching, types of Lyapunov functions, types of stability analysis and techniques of stability analysis. The issues of each aspect corresponding to three categories FMB/PFMB control systems, namely, perfectly, partially and imperfectly matched premises, have been discussed thoroughly to give an idea on how it affects the stability analysis results. In particular, the MFD stability analysis approach, which brings the information of membership functions into the stability conditions, has been explained in details in terms of motivations, techniques, pros and cons. Compared with the MFI stability analysis, which has been employed in majority work in the literature, the MFD stability analysis has demonstrated a great potential to relax the conservativeness and opened a new avenue to the stability analysis of FMB/PFMB control systems. The MFD stability analysis can also provide a fundamental technical and theoretical support to the development of FMB/PFMB control and its applications.

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